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Construction of boundary invariants and the logarithmic singularity of the Bergman kernel

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Introduction

This paper studies Fefferman's program [F3] of expressing the singularity of the Bergman kernel, for smoothly bounded strictly pseudoconvex domains $\Omega \subset \mathbb{C}^n$, in terms of local biholomorphic invariants of the boundary. By [F1], the Bergman kernel on the diagonal $K(z, \overline{z})$ is written in the form

$$K = \varphi\, r^{-n-1} + \psi \log r \quad \text{with} \quad \varphi, \psi \in C^\infty(\overline{\Omega}),$$

where r is a (smooth) defining function of Ω. Recently, Bailey, Eastwood and Graham [BEG], building on Fefferman's earlier work [F3], obtained a full invariant expression of the strong singularity $\varphi\, r^{-n-1}$. The purpose of this paper is to give a full invariant expression of the weak singularity $\psi \log r$.

Fefferman's program is modeled on the heat kernel asymptotics for Riemannian manifolds,

$$K_t(x, x) \sim t^{-n/2} \sum_{j=0}^{\infty} a_j(x)\, t^j \quad \text{as } t \to +0,$$

in which case the coefficients a_j are expressed, by the Weyl invariant theory, in terms of the Riemannian curvature tensor and its covariant derivatives. The Bergman kernel's counterpart of the time variable t is a defining function r of the domain Ω. By [F1] and [BS], the formal singularity of K at a boundary point p is uniquely determined by the Taylor expansion of r at p. Thus one has hope of expressing φ modulo $O^{n+1}(r)$ and ψ modulo $O^\infty(r)$ in terms of local biholomorphic invariants of the boundary, provided r is appropriately chosen. In [F3], Fefferman proposed to find such expressions by reducing the problem to an algebraic one in invariant theory associated with CR geometry, and indeed expressed φ modulo $O^{n-19}(r)$ invariantly by solving the reduced problem partially. The solution in [F3] was then completed in [BEG] to give a full invariant expression of φ modulo $O^{n+1}(r)$, but the reduction is still

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obstructed at finite order so that the procedure does not apply to the log term ψ. We thus modify the invariant-theoretic problem in [F3], [BEG] and solve the modified problem to extend the reduction.

In the heat kernel case, the reduction to the algebraic problem is done by using normal coordinates, and the coefficient functions a_k at a point of reference are $\mathrm{O}(n)$-invariant polynomials in jets of the metric. The CR geometry counterpart of the normal coordinates has been given by Moser [CM]. If $\partial\Omega \in C^\omega$ (real-analytic) then, after a change of local coordinates, $\partial\Omega$ is locally placed in Moser's normal form:

$$N(A): \quad \rho(z,\overline{z}) = 2u - |z'|^2 - \sum_{|\alpha|,|\beta|\geq 2, l\geq 0} A^l_{\alpha\overline{\beta}} z'^\alpha \overline{z}'^\beta v^l = 0, \tag{0.1}$$

where $z' = (z^1, \ldots, z^{n-1})$, $z^n = u + iv$, $A = (A^l_{\alpha\overline{\beta}})$, and the coefficients $A^l_{\alpha\overline{\beta}}$ satisfy trace conditions which are linear (see Section 3). For each $p \in \partial\Omega$, Moser's local coordinate system as above is uniquely determined up to an action of a parabolic subgroup H of $\mathrm{SU}(1,n)$. Thus H-invariant functions of A give rise to local biholomorphic invariants at the point p. Among these invariants, we define *CR invariants of weight* w to be polynomials $I(A)$ in A such that

$$I(\widetilde{A}) = |\det \Phi'(0)|^{-2w/(n+1)} I(A) \tag{0.2}$$

for biholomorphic maps Φ such that $\Phi(0) = 0$ and $\Phi(N(A)) = N(\widetilde{A})$. A CR invariant $I(A)$ defines an assignment, to each strictly pseudoconvex hypersurface $M \in C^\omega$, of a function $I_M \in C^\omega(M)$, which is also called a CR invariant. Here $I_M(p)$, $p \in M$, is given by taking a biholomorphic map such that $\Phi(p) = 0$, $\Phi(M) = N(A)$ and then setting

$$I_M(p) = |\det \Phi'(p)|^{2w/(n+1)} I(A). \tag{0.3}$$

This value is independent of the choice of Φ with $N(A)$ because of (0.2). If $M \in C^\infty$ then (0.3) gives $I_M \in C^\infty(M)$, though a normal form of M can be a formal surface.

The difficulty of the whole problem comes from the ambiguity of the choice of defining functions r, and this has already appeared in the problem for φ, that is, the problem of finding an expression for φ of the form

$$\varphi = \sum_{j=0}^{n} \varphi_j \, r^j + O^{n+1}(r) \quad \text{with} \quad \varphi_j \in C^\infty(\overline{\Omega}), \tag{0.4}$$

such that the boundary value $\varphi_j|_{\partial\Omega}$ is a CR invariant of weight j. Though this expansion looks similar to that of the heat kernel, the situation is much more intricate. It is impossible to choose an exactly invariant defining function r, and thus the extension of CR invariants $\varphi_j|_{\partial\Omega}$ to Ω near $\partial\Omega$, which is crucial, is inevitably approximate. Fefferman [F3] employed an approximately invariant

defining function $r = r^{\mathrm{F}}$, which was constructed in [F2] as a smooth approximate solution to the (complex) Monge-Ampère equation (with zero Dirichlet condition). This defining function is uniquely determined with error of order $n + 2$ along the boundary, and approximately invariant under biholomorphic maps $\Phi\colon \Omega \to \widetilde{\Omega}$ in the sense that

$$\widetilde{r} \circ \Phi = |\det \Phi'|^{2/(n+1)} r + O^{n+2}(r), \tag{0.5}$$

for $r = r^{\mathrm{F}}$ and $\widetilde{r} = \widetilde{r}^{\mathrm{F}}$ associated with Ω and $\widetilde{\Omega}$, respectively. The defining function $r = r^{\mathrm{F}}$ was used by [F3] and [BEG] also in the ambient metric construction of the coefficient functions φ_j explained as follows. Let $g[r]$ be the Lorentz-Kähler metric on $\mathbb{C}^* \times \overline{\Omega} \subset \mathbb{C}^{n+1}$ near $\mathbb{C}^* \times \partial\Omega$ defined by the potential $|z^0|^2 r$ ($z^0 \in \mathbb{C}^*$). Then scalar functions are obtained as complete contractions of tensor products of covariant derivatives of the curvature tensor of $g[r]$. By [F3] and [BEG], such complete contractions generate all CR invariants of weight $\leq n$, and each φ_j in the expansion of φ is realized by linear combinations of these complete contractions.

The approximately invariant defining function $r = r^{\mathrm{F}}$ is too rough in getting an expansion for ψ analogous to that for φ, while there is no hope of making r exactly invariant. Instead, we consider a family $\mathcal{F}_M$ of defining functions of the germ M of $\partial\Omega$ at a point p of reference such that $\mathcal{F}_M$ is invariant under local biholomorphic maps $\Phi\colon M \to \widetilde{M}$, that is, $r \in \mathcal{F}_M$ if and only if $\widetilde{r} \in \mathcal{F}_{\widetilde{M}}$, where $\widetilde{r} \circ \Phi = |\det \Phi'|^{2/(n+1)} r$. We also require that $\mathcal{F}_M$ is parametrized formally by $C^\infty(M)$. More precisely, M is a formal surface, r is a formal function, and $C^\infty(M)$ should be replaced by a space $C^\infty_{\mathrm{formal}}(M)$ of formal power series. If M is in normal form $N(A)$ with $p = 0$, then $f \in C^\infty_{\mathrm{formal}}(M)$ is identified with the Taylor coefficients $C = (C^l_{\alpha\overline{\beta}})$ of $f(z', \overline{z}', v)$ as in (0.1), so that the corresponding $r \in \mathcal{F}_M$ has the parametrization $r = r[A, C]$. Specific construction of $\mathcal{F}_M$ is done by lifting the Monge-Ampère equation to $\mathbb{C}^* \times \Omega$ near $\mathbb{C}^* \times \partial\Omega$ and considering a family of local (or formal) asymptotic solutions, say $\mathcal{F}^{\mathrm{aux}}_M$, which is parametrized by $C^\infty_{\mathrm{formal}}(M)$. This is a refinement of Graham's construction [G2] of asymptotic solutions to the Monge-Ampère equation in Ω. Then, $\mathcal{F}_M$ consists of the smooth parts of elements of $\mathcal{F}^{\mathrm{aux}}_M$, and the parametrization $C^\infty_{\mathrm{formal}}(M) \to \mathcal{F}_M$ for $M = N(A)$ is given by the inverse map of $r \mapsto \partial_\rho^{n+2} r|_{\rho=0}$, which comes from the parametrization of $\mathcal{F}^{\mathrm{aux}}_M$.

Biholomorphic invariance of $\mathcal{F}_M$ gives rise to an extension of the H-action on the normal form coefficients A to that on the pairs (A, C). In fact, a natural generalization of the CR invariant is obtained by considering polynomials $I(A, C)$ in the variables $A^l_{\alpha\overline{\beta}}$ and $C^l_{\alpha\overline{\beta}}$ such that

$$I(\widetilde{A}, \widetilde{C}) = |\det \Phi'(0)|^{-2w/(n+1)} I(A, C)$$

as in (0.2), for biholomorphic maps Φ and $(\widetilde{A}, \widetilde{C})$ satisfying $r[\widetilde{A}, \widetilde{C}] \circ \Phi = |\det \Phi'|^{2/(n+1)} r[A, C]$. Such a polynomial defines an assignment, to each pair (M, r) with $r \in \mathcal{F}_M$, of a function $I[r] \in C^\infty(M)$:

$$I[r](p) = |\det \Phi'(p)|^{2w/(n+1)} I(A, C), \tag{0.6}$$

with Φ as in (0.3) and (A, C) parametrizing $\widetilde{r}$ such that $\widetilde{r} \circ \Phi = |\det \Phi'|^{2/(n+1)} r$. We thus refer to $I(A, C)$ as an *invariant of the pair* (M, r) *of weight* w.

The problem for ψ is then formulated as that of finding an asymptotic expansion of ψ in powers of $r \in \mathcal{F}_{\partial\Omega}$ of the form

$$\psi = \sum_{j=0}^{\infty} \psi_j[r]\, r^j + O^\infty(r) \quad \text{with} \quad \psi_j[r] \in C^\infty(\overline{\Omega}), \tag{0.7}$$

such that each $\psi_j[r]|_{\partial\Omega}$ is an invariant of the pair $(\partial\Omega, r)$ of weight $j + n + 1$. As in the CR invariant case, a class of invariants of the pair $(\partial\Omega, r)$ is obtained by taking the boundary value for linear combinations of complete contractions of tensor products of covariant derivatives of the curvature of the metric $g[r]$. Elements of this class are called *Weyl invariants*. We prove that all invariants of the pair (M, r) are Weyl invariants (see Theorems 4 and 5), so that the expansion (0.7) holds with $\psi_j[r]|_{\partial\Omega}$ given by Weyl invariants of weight $j+n+1$ (see Theorem 1).

A CR invariant $I(A)$ is the same as an invariant of the pair (M, r) which is independent of the parameter C, so that $I(A)$ is a Weyl invariant independent of C (the converse also holds). That is, CR invariants are the same as Weyl invariants independent of the parameter C (see Theorem 2 which follows from Theorems 4 and 5). For Weyl invariants of low weight, it is easy to examine the dependence on C. We have that all Weyl invariants of weight $\le n + 2$ are independent of C (see Theorem 3). This improves the result of [F3] and [BEG] described above by weight 2. If $n = 2$, we have a better estimate (see Theorem 3 again) which is consistent with the results in [HKN2].

Introducing the parameter C was inspired by the work of Graham [G2] on local determination of the asymptotic solution to the Monge-Ampère equation in Ω. He proved approximate invariance, under local biholomorphic maps, of the log term coefficients of the asymptotic solution, and gave a construction of CR invariants of arbitrarily high weight. In our terminology of Weyl invariants, these CR invariants are characterized as complete contractions which contain the Ricci tensor of $g[r]$ (see Remark 5.7 for the precise statement).

This paper is organized as follows. In Section 1, we define the family $\mathcal{F}_M$ of defining functions and state our main results, Theorems 1, 2 and 3. Section 2 is devoted to the construction of the family $\mathcal{F}_M$ and the proof of its biholomorphic invariance. After reviewing the definition of Moser's normal form, we reformulate, in Section 3, CR invariants and invariants of the pair (M, r) as polynomials in (A, C) which are invariant under the action of

H. Then we relate these H-invariant polynomials with those in the variables $R_{i\bar{\jmath}k\bar{l};ab\ldots c}$ on which H acts tensorially, where $R_{i\bar{\jmath}k\bar{l};ab\ldots c}$ are the components of the curvature of $g[r]$ and its covariant derivatives. Using this relation, we reduce our main Theorems 1–3 to the assertion that all invariants of the pair (M, r) are Weyl invariants. This assertion is proved in two steps in Sections 4 and 5. In Section 4, we express all invariants of the pair (M, r) as H-invariant polynomials in $R_{i\bar{\jmath}k\bar{l};ab\ldots c}$. In Section 5, we show that all such H-invariant polynomials come from Weyl invariants, where invariant theory of H in [BEG] is used essentially. In the final Section 6, we study the dependence of Weyl invariants on the parameter C.

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1. Statement of the results

1.1. *Weyl functionals with exact transformation law.* Our concern is a refinement of the ambient metric construction as in [F3], [BEG]. Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded strictly pseudoconvex domain and

$$J(u) = (-1)^n \det \begin{pmatrix} u & u_{\bar{\jmath}} \\ u_i & u_{i\bar{\jmath}} \end{pmatrix}_{1\le i,\, j\le n} \quad \text{where} \quad u_{i\bar{\jmath}} = \partial_{z^i}\partial_{\overline{z}^j} u.$$

In [F3], [BEG], the construction started by choice of a defining function r, with $r > 0$ in Ω, satisfying $J(r) = 1 + O^{n+1}(\partial\Omega)$, where $O^{n+1}(\partial\Omega)$ stands for a term which is smoothly divisible by r^{n+1}. Such an r is unique modulo $O^{n+2}(\partial\Omega)$ and we denote the equivalence class by $\mathcal{F}^{\mathrm{F}}_{\partial\Omega}$. We here consider a subclass $\mathcal{F}_{\partial\Omega}$ of $\mathcal{F}^{\mathrm{F}}_{\partial\Omega}$, which is defined by lifting the (complex) Monge-Ampère equation (with Dirichlet boundary condition)

$$J(u) = 1 \text{ and } u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{1.1}$$

For a function $U(z^0, z)$ on $\mathbb{C}^* \times \overline{\Omega}$, we set

$$J_\#(U) = (-1)^n \det\,(U_{i\bar{\jmath}})_{0\le i,j\le n}$$

and consider a Monge-Ampère equation on $\mathbb{C}^* \times \Omega$:

$$J_\#(U) = |z^0|^{2n} \text{ with } U > 0 \text{ in } \mathbb{C}^* \times \Omega, \text{ and } \quad U = 0 \text{ on } \mathbb{C}^* \times \partial\Omega. \tag{1.2}$$

If U is written as $U(z^0, z) = |z^0|^2 u(z)$ with a function $u(z)$ on Ω, then (1.2) is reduced to (1.1) because $J_\#(U) = |z^0|^{2n} J(u)$. In [G2], Graham fixed $r \in \mathcal{F}^{\mathrm{F}}_{\partial\Omega}$ arbitrarily and constructed asymptotic solutions u^{G} to (1.1) of the form

$$u^{\mathrm{G}} = r \sum_{k=0}^{\infty} \eta^{\mathrm{G}}_k \left(r^{n+1} \log r\right)^k \quad \text{with} \quad \eta^{\mathrm{G}}_k \in C^\infty(\overline{\Omega}), \tag{1.3}$$

which are parametrized by the space $C^\infty(\partial\Omega)$ of initial data (see Remark 1.1 below). Then $U^{\mathrm{G}} = |z^0|^2 u^{\mathrm{G}}$ are asymptotic solutions to (1.2). We here modify these asymptotic solutions and consider another class of asymptotic solutions of the form

$$U = r_\# + r_\# \sum_{k=1}^{\infty} \eta_k \left(r^{n+1} \log r_\#\right)^k \quad \text{with} \quad \eta_k \in C^\infty(\overline{\Omega}), \tag{1.4}$$

again parametrized by $C^\infty(\partial\Omega)$, where $r_\# = |z^0|^2 r$ with $r \in \mathcal{F}^{\mathrm{F}}_{\partial\Omega}$. It should be emphasized that r is not prescribed but determined by U. Note also that U is not of the form $|z^0|^2 u$ because $\log r_\#$ is not homogeneous in z^0. We call r in (1.4) the *smooth part* of U and define $\mathcal{F}_{\partial\Omega}$ to be the totality of the smooth parts of asymptotic solutions to (1.2) for $\partial\Omega$.

We identify two asymptotic solutions of the form (1.4) if the corresponding functions r and η_k agree to infinite order along $\partial\Omega$. Then the unique existence of the asymptotic solution U as in (1.4) holds once the initial data are given in $C^\infty(\partial\Omega)$.

Propostition 1. *Let X be a real vector field on $\overline{\Omega}$ which is transversal to $\partial\Omega$. Then for any $a \in C^\infty(\partial\Omega)$, there exists a unique asymptotic solution U to* (1.2) *for $\partial\Omega$ such that the smooth part r satisfies*

$$X^{n+2} r|_{\partial\Omega} = a. \tag{1.5}$$

The lifted Monge-Ampère equation (1.2) and the asymptotic solutions of the form (1.4) are introduced in order to obtain the following exact transformation law for the smooth part r.

Propostition 2. *Let $\Phi\colon \Omega \to \widetilde{\Omega}$ be a biholomorphic map. Then $r \in \mathcal{F}_{\partial\Omega}$ if and only if $\widetilde{r} \in \mathcal{F}_{\partial\widetilde{\Omega}}$, where $\widetilde{r}$ is given by*

$$\widetilde{r} \circ \Phi = |\det \Phi'|^{2/(n+1)} r. \tag{1.6}$$

Here $\det \Phi'$ *is the holomorphic Jacobian of* Φ.

Remark 1.1. For u^{G} in (1.3), $\eta_0^{\mathrm{G}} = 1 + O^{n+1}(\partial\Omega)$ holds. To make u^{G} unique, Graham [G2] used the boundary value of $(\eta_0^{\mathrm{G}} - 1)/r^{n+1}|_{\partial\Omega}$ as the initial data $a \in C^\infty(\partial\Omega)$, where r is arbitrarily fixed. It is also possible to make u^{G} unique by requiring $\eta_0^{\mathrm{G}} = 1$ in (1.3), in which case r is determined by u^{G} (cf. Lemma 2.3). Then we may write $r = r[u^{\mathrm{G}}]$ and consider the totality of these, say $\mathcal{F}^{\mathrm{G}}_{\partial\Omega}$. However, $\mathcal{F}^{\mathrm{G}}_{\partial\Omega}$ does not satisfy the transformation law (1.6) in Proposition 2; it is not the case that every $\widetilde{r} = r[\widetilde{u}^{\mathrm{G}}] \in \mathcal{F}^{\mathrm{G}}_{\partial\widetilde{\Omega}}$ is given by (1.6) with some $r = r[u^{\mathrm{G}}] \in \mathcal{F}^{\mathrm{G}}_{\partial\Omega}$. Though the proof requires some preparation (cf. Remark 4.8), this is roughly seen by the fact that (1.6) implies $(\log \widetilde{r}) \circ \Phi = \log r + \log |\det \Phi'|^{2/(n+1)}$, which destroys the condition $\widetilde{\eta}_0^{\mathrm{G}} = \eta_0^{\mathrm{G}}[\widetilde{u}^{\mathrm{G}}] = 1$ (cf. subsection 2.1).

For each defining function $r \in \mathcal{F}_{\partial\Omega}$, we define a Lorentz-Kähler metric

$$g[r] = \sum_{i,j=0}^{n} \frac{\partial^2 r_\#}{\partial z^i \partial \overline{z}^j}\, dz^i\, d\overline{z}^j \quad \text{on } \mathbb{C}^* \times \overline{\Omega} \text{ near } \mathbb{C}^* \times \partial\Omega.$$

We call this metric $g = g[r]$ an *ambient metric* associated with $\partial\Omega$. From the ambient metric, we construct scalar functions as follows. Let R denote the curvature tensor of g and $R^{(p,q)} = \overline{\nabla}^{q-2}\nabla^{p-2}R$ the successive covariant derivatives, where ∇ (resp. $\overline{\nabla}$) stands for the covariant differentiation of type $(1,0)$ (resp. $(0,1)$). Then a complete contraction of the form

$$W_\# = \mathrm{contr}(R^{(p_1,q_1)} \otimes \cdots \otimes R^{(p_d,q_d)}) \tag{1.7}$$

gives rise to a function $W_\#[r]$ on $\mathbb{C}^* \times \overline{\Omega}$ near $\mathbb{C}^* \times \partial\Omega$ once $r \in \mathcal{F}_{\partial\Omega}$ is specified. Here contractions are taken with respect to the ambient metric for some pairing of holomorphic and antiholomorphic indices. The *weight* of $W_\#$ is defined by $w = -d + \sum_{j=1}^{d}(p_j + q_j)/2$, which is an integer because $\sum p_j = \sum q_j$ holds. By a *Weyl polynomial*, we mean a linear combination of $W_\#$ of the form (1.7) of homogeneous weight. A Weyl polynomial gives a functional for the pair $(\partial\Omega, r)$ which satisfies a transformation law under biholomorphic maps. To state this precisely, we make the following definition.

Definition 1.2. A Weyl polynomial $W_\#$ of weight w assigns, to each pair $(\partial\Omega, r)$ with $r \in \mathcal{F}_{\partial\Omega}$, a function $W[r] = W_\#[r]|_{z^0=1}$ on $\overline{\Omega}$ near $\partial\Omega$. We call this assignment $W\colon r \mapsto W[r]$ a *Weyl functional of weight* w associated with $W_\#$.

PROPOSTITION 3. *Let W be a Weyl functional of weight w. Then, for r and $\widetilde{r}$ as in* (1.6),

$$W[\widetilde{r}] \circ \Phi = |\det \Phi'|^{-2w/(n+1)} W[r]. \tag{1.8}$$

We refer to the relation (1.8) as a transformation law of weight w for W.

Remark 1.3. Without change of the proof, Proposition 1 can be localized near a boundary point p. That is, we may replace $\partial\Omega$ by a germ M of $\partial\Omega$ at p or a formal surface, and r, η_k, a by germs of smooth functions or formal power series about p. Then $\mathcal{F}_{\partial\Omega}$ is a sheaf $(\mathcal{F}_{p,\overline{\Omega}})_{p\in\partial\Omega}$. Abusing notation, we write $\mathcal{F}_M$ in place of $\mathcal{F}_{p,\overline{\Omega}}$. Then Propositions 2 and 3 also have localization, where Φ is a (formal) biholomorphic map such that $\Phi(M) = \widetilde{M}$ with $\widetilde{M}$ associated to $\widetilde{\Omega}$.

1.2. *Invariant expansion of the Bergman kernel.* For each $r \in \mathcal{F}_{\partial\Omega}$, we write the asymptotic expansion of the Bergman kernel of Ω on the diagonal $K(z) = K(z, \overline{z})$ as follows:

$$K = \varphi[r]\, r^{-n-1} + \psi[r] \log r \quad \text{with} \quad \varphi[r], \psi[r] \in C^\infty(\overline{\Omega}), \tag{1.9}$$

where we regard $\varphi = \varphi[r]$ and $\psi = \psi[r]$ as functionals of the pair $(\partial\Omega, r)$. Note that $\varphi[r] \bmod O^{n+1}(\partial\Omega)$ and $\psi[r] \bmod O^{\infty}(\partial\Omega)$ are independent of the choice of r. In our first main theorem, we express these functionals in terms of Weyl functionals.

Theorem 1. *For $n \geq 2$, there exist Weyl functionals W_k of weight k for $k = 0, 1, 2, \ldots$ such that*

$$\varphi[r] = \sum_{k=0}^{n} W_k[r]\, r^k + O^{n+1}(\partial\Omega), \tag{1.10}$$

$$\psi[r] = \sum_{k=0}^{\infty} W_{k+n+1}[r]\, r^k + O^{\infty}(\partial\Omega), \tag{1.11}$$

for any strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ and any $r \in \mathcal{F}_{\partial\Omega}$. Here (1.11) *means that $\psi[r] = \sum_{k=0}^{m} W_{k+n+1}[r]\, r^k + O^{m+1}(\partial\Omega)$ for any $m \geq 0$.*

The expansion (1.10) has been obtained in [F3] and [BEG], where r is any defining function satisfying $J(r) = 1 + O^{n+1}(\partial\Omega)$. This condition is fulfilled by our $r \in \mathcal{F}_{\partial\Omega}$.

1.3. CR *invariants in terms of Weyl invariants.* Suppose $\partial\Omega$ is in Moser's normal form (0.1) near 0. With the real coordinates $(z', \overline{z}', v, \rho)$, we write the Taylor series about 0 of $\partial_\rho^{n+2} r|_{\rho=0}$ for $r \in \mathcal{F}_{\partial\Omega}$ as

$$\partial_\rho^{n+2} r|_{\rho=0} = \sum_{|\alpha|, |\beta|, l \geq 0} C^l_{\alpha\overline{\beta}} z'^\alpha \overline{z}'^\beta v^l. \tag{1.12}$$

Then for a Weyl functional W, the value $W[r](0)$ is expressed as a universal polynomial $I_W(A, C)$ in the variables $A^l_{\alpha\overline{\beta}}, C^l_{\alpha\overline{\beta}}$. We call this polynomial a *Weyl invariant* and say that I_W is *$\mathcal{C}$-independent* if it is independent of the variables $C^l_{\alpha\overline{\beta}}$. Our second main theorem asserts that $\mathcal{C}$-independent Weyl invariants give all CR invariants.

Theorem 2. *All $\mathcal{C}$-independent Weyl invariants are* CR *invariants, and vice versa.*

It is not easy to determine which Weyl invariant I_W is $\mathcal{C}$-independent when the weight w of I_W is high. If $w \leq n+2$ (resp. $w \leq 5$) for $n \geq 3$ (resp. $n = 2$), then we can show that W is $\mathcal{C}$-independent (Proposition 6.1). Thus Theorem 2 yields:

Theorem 3. *For weight $\leq n+2$, all Weyl invariants are* CR *invariants and vice versa. Moreover, for $n = 2$, the same is true for weight ≤ 5.*

In this theorem, the restriction on weight is optimal. In fact, there exists a $\mathcal{C}$-dependent Weyl invariant of weight $n+3$, or weight 6 when $n=2$ (Proposition 6.1). Thus, to obtain a complete list of CR invariants for this or higher weights, one really needs to select $\mathcal{C}$-independent Weyl invariants. This is a problem yet to be studied.

Remark 1.4. In the introduction, we defined a Weyl invariant to be the boundary value of a Weyl functional. This definition is consistent with the one given here as a polynomial $I_W(A, C)$. In fact, $I_W(A, C)$ defines via (0.6) an assignment, to each pair $(\partial\Omega, r)$, of a function $I_W[r] \in C^\infty(\partial\Omega)$ which coincides with $W[r]|_{\partial\Omega}$. This corresponds to the identification of a CR invariant $I(A)$ with the boundary functional induced by $I(A)$.

2. Asymptotic solutions of the complex Monge-Ampère equation

In this section we prove Propositions 1, 2 and 3. We first assume Proposition 1 and prove Propositions 2 and 3, the transformation laws of $\mathcal{F}_{\partial\Omega}$ and Weyl functionals.

2.1. *Proof of Propositions* 2 *and* 3. For a biholomorphic map $\Phi: \Omega \to \widetilde{\Omega}$, we define the lift $\Phi_\#: \mathbb{C}^* \times \Omega \to \mathbb{C}^* \times \widetilde{\Omega}$ by

$$\Phi_\#(z^0, z) = \left(z^0 \cdot [\det \Phi'(z)]^{-1/(n+1)}, \Phi(z)\right), \tag{2.1}$$

where a branch of $[\det \Phi']^{-1/(n+1)}$ is arbitrarily chosen. Then $\det \Phi'_\#(z^0, z) = [\det \Phi'(z)]^{n/(n+1)}$, so that

$$(|z^0|^{-2n} \det(\widetilde{U}_{i\bar{\jmath}})) \circ \Phi_\# = |z^0|^{-2n} \det((\widetilde{U} \circ \Phi_\#)_{i\bar{\jmath}})$$

for any function $\widetilde{U}$ on $\mathbb{C}^* \times \widetilde{\Omega}$. In particular, if U is an asymptotic solution of (1.2) for $\partial\Omega$, so is $\widetilde{U} = U \circ \Phi_\#^{-1}$ for $\partial\widetilde{\Omega}$. The expansion of $\widetilde{U}$ is given by

$$\widetilde{U} = \widetilde{r}_\# + \widetilde{r}_\# \sum_{k=1}^{\infty} \widetilde{\eta}_k \, (\widetilde{r}^{\,n+1} \log \widetilde{r}_\#)^k,$$

where $\widetilde{r} \circ \Phi = |\det \Phi'|^{2/(n+1)} r$ and $\widetilde{\eta}_k \circ \Phi = |\det \Phi'|^{-2k} \eta_k$. It follows that $\widetilde{r}$ is the smooth part of $\widetilde{U}$ if and only if $r = |\det \Phi'|^{-2/(n+1)} \widetilde{r} \circ \Phi$ is the smooth part of $U = \widetilde{U} \circ \Phi_\#$. This proves Proposition 2.

We next prove Proposition 3. Writing the transformation law (1.6) as $\widetilde{r}_\# \circ \Phi_\# = r_\#$ and applying $\partial\overline{\partial}$ to it, we see that $\Phi_\#: (\mathbb{C}^* \times \Omega, g[r]) \to (\mathbb{C}^* \times \widetilde{\Omega}, g[\widetilde{r}])$ is an isometry. If $W_\#$ is a Weyl polynomial of weight w, then

$$W_\#[\widetilde{r}] \circ \Phi_\# = W_\#[r], \tag{2.2}$$

while the homogeneity of the ambient metric in z^0 implies

$$W_{\#}[r] = |z^0|^{-2w} W[r].$$

Thus (2.2) is rewritten as (1.8), and Proposition 3 is proved.

2.2. *Proof of Proposition* 1. We fix a defining function ρ satisfying $J(\rho) = 1 + O^{n+1}(\partial\Omega)$ and introduce a nonlinear differential operator for functions f on $\mathbb{C}^* \times \Omega$:

$$\mathcal{M}(f) = \det(U_{i\overline{j}})/\det((\rho_{\#})_{i\overline{j}}) \quad \text{with } U = \rho_{\#}\,(1+f).$$

Then $J_{\#}(U) = |z^0|^{2n}$ is written as

$$\mathcal{M}(f) = J(\rho)^{-1}. \tag{2.3}$$

If U is a series of the form (1.4), then f admits an expansion

$$f = \sum_{k=0}^{\infty} \eta_k (\rho^{n+1} \log \rho_{\#})^k, \quad \text{where } \eta_k \in C^\infty(\overline{\Omega}).$$

Denoting by $\mathcal{A}$ the space of all formal series of this form, we shall construct solutions to (2.3) in $\mathcal{A}$.

We first study the degeneracy of the equation (2.3) at the surface $\mathbb{C}^* \times \partial\Omega$. Following [G2], we use a local frame $Z_0, \ldots, Z_n$ of $T^{(1,0)}(\mathbb{C}^* \times \overline{\Omega})$ near $\mathbb{C}^* \times \partial\Omega$ satisfying:

(1) $Z_0 = z^0(\partial/\partial z^0)$;

(2) $Z_1, \ldots, Z_{n-1}$ are orthonormal vector fields on $\overline{\Omega}$ with respect to the Levi form $\partial\overline{\partial}\rho$ such that $Z_j \rho = 0$;

(3) Z_n is a vector field on $\overline{\Omega}$ such that $Z_n \,\lrcorner\, \partial\overline{\partial}\rho = \gamma\, \overline{\partial}\rho$ for some $\gamma \in C^\infty(\overline{\Omega})$, $N\rho = 1$ and $T\rho = 0$, where $N = \mathrm{Re} Z_n$, $T = \mathrm{Im} Z_n$.

Using this frame, we introduce a ring $\mathcal{P}_{\partial\Omega}$ of differential operators on $\mathbb{C}^* \times \overline{\Omega}$ that are written as polynomials of $Z_0, \ldots, Z_{n-1}, \overline{Z}_0, \ldots, \overline{Z}_{n-1}, T, \rho N$ with coefficients in $C^\infty(\overline{\Omega}, \mathbb{C})$, the space of complex-valued smooth functions on $\overline{\Omega}$. In other words, $\mathcal{P}_{\partial\Omega}$ is a ring generated by $Z_0, \overline{Z}_0$ and totally characteristic operators on $\overline{\Omega}$ in the sense of [LM]. We first express $\mathcal{M}$ as a nonlinear operator generated by $\mathcal{P}_{\partial\Omega}$.

Lemma 2.1. *Let* $E = -(\rho N + 1)(\rho N - 2Z_0 - n - 1)$. *Then,*

$$\mathcal{M}(f) = 1 + Ef + \rho P_0 f + Q(P_1 f, \ldots, P_l f) \quad \text{for } f \in \mathcal{A}, \tag{2.4}$$

where $P_0, P_1, \ldots, P_l \in \mathcal{P}_{\partial\Omega}$, *and* Q *is a polynomial without constant and linear terms.*

Proof. Taking the dual frame $\omega^0, \dots, \omega^n$ of $Z_0, \dots, Z_n$, we set $\theta^j = z^0 \omega^j$. Then, the conditions (1)–(3) imply $\theta^0 = dz^0$, $\theta^n = z^0 \partial\rho$ and

$$\partial\overline{\partial}\rho_\# = \rho\theta^0 \wedge \overline{\theta^0} + \theta^0 \wedge \overline{\theta^n} + \theta^n \wedge \overline{\theta^0} - \sum_{i=1}^{n-1} \theta^i \wedge \overline{\theta^i} + \gamma\theta^n \wedge \overline{\theta^n}. \tag{2.5}$$

Using the coframe $\theta^0, \dots, \theta^n$, we define a Hermitian matrix $A(f) = (A_{i\overline{j}}(f))$ by

$$\partial\overline{\partial}\left(\rho_\# (1+f)\right) = \sum_{i,\, j=0}^{n} A_{i\overline{j}}(f)\theta^i \wedge \overline{\theta^j},$$

so that $\mathcal{M}(f) = \det A(f)/\det A(0)$ holds. Let us compute $A(f)$. First,

$$\partial\overline{\partial}\left(\rho_\# (1+f)\right) = (1+f)\partial\overline{\partial}\,\rho_\# + \partial f \wedge \overline{\partial}\rho_\# + \partial\rho_\# \wedge \overline{\partial} f + \rho_\# \partial\overline{\partial}\, f.$$

For the first term on the right-hand side, we use (2.5). The second and the third terms are respectively given by

$$\partial f \wedge \overline{\partial}\rho_\# = \sum_{j=0}^{n} Z_j f\, \theta^j \wedge (\rho\,\overline{\theta^0} + \overline{\theta^n})$$

and its complex conjugate. Finally, for the last term,

$$\rho_\# \partial\overline{\partial} f = \rho \sum_{i,j=0}^{n} (Z_i \overline{Z_j} + E_{i\overline{j}}) f\, \theta^i \wedge \overline{\theta^j} + \rho_\# N f \partial\overline{\partial}\rho,$$

where $E_{i\overline{j}} \in \mathcal{P}_{\partial\Omega}$ with $E_{0\overline{j}} = E_{j\overline{0}} = 0$ for any j. Therefore, $A(f)$ modulo functions of the form $\rho P f$, $P \in \mathcal{P}_{\partial\Omega}$, is given by

$$\begin{pmatrix} \rho & 0 & 1 + P_{0\overline{n}} f \\ 0 & -\delta_{i\overline{j}}(1 + f + \rho N f) & * \\ 1 + P_{n\overline{0}} f & * & \gamma + P_{n\overline{n}} f \end{pmatrix},$$

where $*$ stands for a function of the form $Pf, P \in \mathcal{P}_{\partial\Omega}$, and

$$\begin{aligned} P_{0\overline{n}} &= \overline{P_{n\overline{0}}} = 1 + \rho\overline{Z_n} + Z_0 + \rho Z_0 \overline{Z_n}, \\ P_{n\overline{n}} &= \gamma + Z_n + \overline{Z_n} + \rho Z_n \overline{Z_n} + \gamma\rho N \\ &= \rho N^2 + 2N \mod \mathcal{P}_{\partial\Omega}. \end{aligned}$$

Let $B(f)$ denote the matrix obtained from $A(f)$ by dividing the first column by ρ and multiplying the last row by ρ. Then $B(f)$ modulo functions of the form $\rho P f, P \in \mathcal{P}_{\partial\Omega}$, is given by

$$\begin{pmatrix} 1 & 0 & 1 + P_{0\overline{n}} f \\ * & -\delta_{i\overline{j}}(1 + f + \rho N f) & * \\ 1 + P_{n\overline{0}} f & 0 & \gamma\rho + \rho^2 N^2 f + 2\rho N f \end{pmatrix}.$$

Noting that $\det A(f) = \det B(f)$, we get

$$\begin{aligned}\mathcal{M}(f) &= 1 - \rho^2 N^2 f - 2\rho N f + (n-1)(1+\rho N) f \\ &\quad + P_{n\overline{0}} f + P_{0\overline{n}} f + \rho P_0 f + Q(P_1 f, \dots, P_l f).\end{aligned}$$

Using $Z_0 f = \overline{Z_0} f$, we obtain (2.4). □

To construct solutions to (2.3) inductively, we introduce a filtration

$$\mathcal{A} = \mathcal{A}_0 \supset \mathcal{A}_1 \supset \mathcal{A}_2 \supset \cdots,$$

where $\mathcal{A}_s$ denotes the space of all asymptotic series in $\mathcal{A}$ of the form

$$\rho^s \sum_{k=0}^{\infty} \alpha_k (\log \rho_\#)^k \quad \text{with} \quad \alpha_k \in C^\infty(\overline{\Omega}).$$

This filtration makes $\mathcal{A}$ a filtered ring which is preserved by the action of $\mathcal{P}_{\partial\Omega}$. That is, $\mathcal{A}_j \mathcal{A}_k \subset \mathcal{A}_{j+k}$ and $Pf \in \mathcal{A}_j$ for each $(P, f) \in \mathcal{P}_{\partial\Omega} \times \mathcal{A}_j$. Hence (2.4) yields $\mathcal{M}(f+g) = \mathcal{M}(f) + \mathcal{A}_s$ for any $g \in \mathcal{A}_s$. In particular, if $f \in \mathcal{A}$ is a solution to the equation

$$\mathcal{M}(f) = J(\rho)^{-1} \mod \mathcal{A}_{s+1}, \tag{2.3$_s$}$$

so is $\widetilde{f} = f + g$ for any $g \in \mathcal{A}_{s+1}$. We shall show that this equation admits a unique solution modulo $\mathcal{A}_{s+1}$ if an initial condition corresponding to (1.5) is imposed.

Lemma 2.2. (i) *An asymptotic series $f \in \mathcal{A}$ satisfies $(2.3)_n$ if and only if $f \in \mathcal{A}_{n+1}$.*

(ii) *Let $s \geq n+1$. Then, for any $a \in C^\infty(\partial\Omega)$, the equation $(2.3)_s$ admits a solution f_s, which is unique modulo $\mathcal{A}_{s+1}$ under the condition*

$$\eta_0 = a\rho^{n+1} + O^{n+2}(\partial\Omega). \tag{2.6}$$

Proof. Since $f \in \mathcal{A}$ satisfies $f = \eta_0 \mod \mathcal{A}_{n+1}$, it follows that $\mathcal{M}(f) = \mathcal{M}(\eta_0) + \mathcal{A}_{n+1}$. Thus, recalling $\mathcal{M}(\eta_0) = J(\rho(1+\eta_0))/J(\rho)$, we see that $(2.3)_n$ is reduced to

$$J(\rho(1+\eta_0)) = 1 + O^{n+1}(\partial\Omega).$$

This is satisfied if and only if $\eta_0 = O^{n+1}(\partial\Omega)$, which is equivalent to $f \in \mathcal{A}_{n+1}$. Thus (i) is proved.

To prove (ii), we first consider $(2.3)_s$ for $s = n+1$. If $f \in \mathcal{A}_{n+1}$, then $\mathcal{M}(f) = 1 + Ef + \mathcal{A}_{n+2}$. Thus $(2.3)_{n+1}$ is equivalent to

$$Ef = J(\rho)^{-1} - 1 \mod \mathcal{A}_{n+2}. \tag{2.7}$$

Writing $f = \rho^{n+1}(\alpha_0 + \alpha_1 \log\rho_\#) \bmod \mathcal{A}_{n+2}$, we have $Ef = (n+2)\alpha_1 \rho^{n+1} + \mathcal{A}_{n+2}$. Hence, (2.7) holds if and only if $(n+2)\alpha_1 = (J(\rho)^{-1}-1)\rho^{-n-1} + O(\partial\Omega)$. Noting that $\alpha_0|_{\partial\Omega}$ is determined by (2.6), we get the unique existence of f_{n+1} modulo $\mathcal{A}_{n+2}$.

For $s > n+1$, we construct f_s by induction on s. Assume that f_{s-1} exists uniquely modulo $\mathcal{A}_s$. Then we have $\mathcal{M}(f_{s-1}+g) = \mathcal{M}(f_{s-1}) + Eg + \mathcal{A}_{s+1}$ for $g \in \mathcal{A}_s$, so that $(2.3)_s$ is reduced to

$$E[g]_s = [h]_s \quad \text{with} \quad h = J(\rho)^{-1} - \mathcal{M}(f_{s-1}) \in \mathcal{A}_s, \tag{2.8}$$

where $[g]_s$ and $[h]_s$ stand for the cosets in $\mathcal{A}_s/\mathcal{A}_{s+1}$. To solve this equation, we introduce a filtration of $\mathcal{A}_s/\mathcal{A}_{s+1}$:

$$\mathcal{A}_s/\mathcal{A}_{s+1} = \mathcal{A}_s^{(l)} \supset \mathcal{A}_s^{(l-1)} \supset \cdots \supset \mathcal{A}_s^{(0)} \supset \mathcal{A}_s^{(-1)} = \{0\},$$

where $l = [s/(n+1)]$ and

$$\mathcal{A}_s^{(t)} = \left\{ [g]_s \in \mathcal{A}_s/\mathcal{A}_{s+1} : g = \sum_{k=0}^{t} \eta_k (\rho^{n+1}\log\rho_\#)^k \in \mathcal{A}_s \right\}.$$

Clearly, $\rho N \mathcal{A}_s^{(t)} \subset \mathcal{A}_s^{(t)}$ and $Z_0 \mathcal{A}_s^{(t)} \subset \mathcal{A}_s^{(t-1)}$. Consequently, if we write $[g]_s \in \mathcal{A}_s^{(m)}$ as $[g]_s = [\alpha_m \rho^s (\log\rho_\#)^m]_s + \mathcal{A}_s^{(m-1)}$, then

$$E[g]_s = I(s)[\alpha_m \rho^s (\log\rho_\#)^m]_s + \mathcal{A}_s^{(m-1)},$$

where $I(x) = -(x+1)(x-n-1)$. Thus, setting $F = 1 - I(s)^{-1}E$, we have $F[g]_s \in \mathcal{A}_s^{(m-1)}$ so that $F\mathcal{A}_s^{(m)} \subset \mathcal{A}_s^{(m-1)}$. In particular, $F^l = 0$ on $\mathcal{A}_s^{(l)}$. Since $E = I(s)(1-F)$, the linear operator $L = I(s)^{-1}\sum_{k=0}^{l-1} F^k$ satisfies $LE = EL = \mathrm{id}$ on $\mathcal{A}_s^{(l)}$. Therefore, (2.8) admits a unique solution $[g]_s$, which gives a unique solution $f_s = f_{s-1} + g$ modulo $\mathcal{A}_{s+1}$ of $(2.3)_s$. □

The unique solution of (2.3) with the condition (2.6) is obtained by taking the limit of f_s as $s \to \infty$. More precisely, we argue as follows. For $a \in C^\infty(\partial\Omega)$, we take a sequence $\{f_s\}$ in Lemma 2.2, and write $f_s = \sum \eta_k^{(s)} (\rho^{n+1}\log\rho_\#)^k$. Then the uniqueness of f_s mod $\mathcal{A}_{s+1}$ yields $\eta_k^{(s+1)} = \eta_k^{(s)} \bmod O^{s-k(n+1)}(\partial\Omega)$. This implies the existence of $\eta_k \in C^\infty(\overline{\Omega})$ such that

$$\eta_k = \eta_k^{(s)} \bmod O^{s-k(n+1)}(\partial\Omega)$$

for any s. Therefore, the formal series $f = \sum_{k=0}^{\infty} \eta_k (\rho^{n+1}\log\rho_\#)^k$ satisfies $\mathcal{M}(f) = J(\rho)^{-1}$ and (2.6). The uniqueness follows from that for each $(2.3)_s$.

We have constructed a solution $f \in \mathcal{A}$ of (2.3) and hence obtained a formal series

$$U = \rho_\#(1+f) = \rho_\# + \rho_\# \sum_{k=0}^{\infty} \eta_k (\rho^{n+1}\log\rho_\#)^k, \tag{2.9}$$

which solves (1.2) to infinite order along $\mathbb{C}^* \times \partial\Omega$. In general, the series (2.9) is not in the form (1.4) because η_0 may not vanish. We next construct a unique defining function r such that U is written in the form (1.4). In the following, we write $f = g \bmod O^\infty(\partial\Omega)$ if $f - g$ vanishes to infinite order along $\partial\Omega$.

LEMMA 2.3. *Let $f \in \mathcal{A}_{n+1}$. Then there exists a unique defining function $r \bmod O^\infty(\partial\Omega)$ such that $U = \rho_\#(1+f)$ is written in the form* (1.4).

Proof. Starting from $r_1 = \rho$, we define a sequence of defining functions r_s, $s = 1, 2, \ldots$, by setting $r_{s+1} = r_s(1 + \eta_{s,0})$, where $\eta_{s,0}$ is the coefficient in the expansion $U = r_{s\#} + r_{s\#} \sum_{k=0}^\infty \eta_{s,k}(r_s^{n+1} \log r_{s\#})^k$. It then follows from $\log(r_{s+1\#}) = \log(r_{s\#}) + O^{s(n+1)}(\partial\Omega)$ that $\eta_{s,0} = O^{s(n+1)}(\partial\Omega)$, so that $r_{s+1} = r_s + O^{s(n+1)+1}(\partial\Omega)$. We can then construct a defining function r such that $r = r_s \bmod O^{s(n+1)+1}(\partial\Omega)$ for any s. With this r, the series U is written as $U = r_\# + r_\# \sum_{k=1}^\infty \eta_k (r^{n+1} \log r_\#)^k$.

Let us next prove the uniqueness of r. We take another defining function $\widetilde{r}$ with the required property and write $U = \widetilde{r}_\# \sum_{k=0}^\infty \widetilde{\eta}_k (\widetilde{r}^{n+1} \log \widetilde{r}_\#)^k$. Setting $\phi = r/\widetilde{r} \in C^\infty(\overline{\Omega})$, we then have $\widetilde{\eta}_0 = \phi(1 + \sum_{k=1}^\infty \eta_k(\rho^{n+1}\log\phi)^k)$. Since $\widetilde{\eta}_0 = 1$,

$$\frac{1}{\phi} = 1 + \sum_{k=1}^\infty \eta_k (r^{n+1} \log \phi)^k. \tag{2.10}$$

This implies that if $\phi = 1 + O^m(\partial\Omega)$ then $\phi = 1 + O^{m+n+1}(\partial\Omega)$. Therefore, $\phi = 1 \bmod O^\infty(\partial\Omega)$; that is, $\widetilde{r} = r \bmod O^\infty(\partial\Omega)$. □

We next examine the relation between the conditions (1.5) and (2.6). Writing $U = \rho_\#(1+f)$ in the form (1.4), we have

$$r = \rho + \eta_0\, \rho + O^{2(n+1)}(\partial\Omega) = \rho + \rho^{n+2} a + O^{n+3}(\partial\Omega).$$

Applying X^{n+2}, we get

$$X^{n+2} r = X^{n+2}\rho + (n+2)!\,(X\rho)^{n+2} a + O(\partial\Omega).$$

Since X is transversal to $\partial\Omega$, that is, $X\rho|_{\partial\Omega} \neq 0$, it follows that specifying $X^{n+2} r|_{\partial\Omega}$ is equivalent to specifying a in (2.6) when ρ is prescribed. Therefore, f and thus $U = \rho_\#(1+f)$ are uniquely determined by the condition (1.5). This completes the proof of the first statement of Proposition 1.

It remains to prove $r \in \mathcal{F}^{\mathrm{F}}_{\partial\Omega}$; that is, $J(r) = 1 + O^{n+1}(\partial\Omega)$. If we write $U = r_\#(1+f)$ then $\mathcal{M}(f) = J(r)^{-1}$, where $\mathcal{M}$ is defined with respect to $\rho = r$. On the other hand, we have by Lemma 2.2, (i) that $f \in \mathcal{A}_{n+1}$ and thus $\mathcal{M}(f) = \mathcal{M}(0) = 1 \bmod \mathcal{A}_{n+1}$. Therefore, $J(r)^{-1} = 1 \bmod \mathcal{A}_{n+1}$; that is, $J(r) = 1 + O^{n+1}(\partial\Omega)$.

3. Reformulation of the main theorems

3.1. *A group action characterizing* CR *invariants.* We first recall the definition and basic properties of Moser's normal form [CM]. A real-analytic hypersurface $M \subset \mathbb{C}^n$ is said to be in *normal form* if it admits a defining function of the form

$$\rho = 2u - |z'|^2 - \sum_{|\alpha|,|\beta|\geq 2,\, l\geq 0} A^l_{\alpha\overline{\beta}}\, z'^{\alpha}\overline{z}'^{\beta} v^l, \tag{3.1}$$

where the coefficients $(A^l_{\alpha\overline{\beta}})$ satisfy the following three conditions: (N1) For each p, $q \geq 2$ and $l \geq 0$, $\mathbf{A}^l_{p\overline{q}} = (A^l_{\alpha\overline{\beta}})_{|\alpha|=p,\,|\beta|=q}$ is a bisymmetric tensor of type (p,q) on $\mathbb{C}^{n-1}$; (N2) $A^l_{\alpha\overline{\beta}} = \overline{A^l_{\beta\overline{\alpha}}}$; (N3) $\operatorname{tr}\mathbf{A}^l_{2\overline{2}} = 0$, $\operatorname{tr}^2\mathbf{A}^l_{2\overline{3}} = 0$, $\operatorname{tr}^3\mathbf{A}^l_{3\overline{3}} = 0$; here tr denotes the usual tensorial trace with respect to $\delta^{i\overline{j}}$. We denote this surface by $N(A)$ with $A = (A^l_{\alpha\overline{\beta}})$. In particular, $N(0)$ is the hyperquadric $2\,u = |z'|^2$.

For any real-analytic strictly pseudoconvex surface M and $p \in M$, there exist holomorphic local coordinates near p such that M is in normal form and $p = 0$. Moreover, if M is tangent to the hyperquadric to the second order at 0, a local coordinates change $S(z) = w$ for which $S(M)$ is in normal form is unique under the normalization

$$S(0) = 0, \quad S'(0) = \mathrm{id}, \quad \operatorname{Im}\frac{\partial^2 w^n}{\partial (z^n)^2}(0) = 0. \tag{3.2}$$

Even if M is not real-analytic but merely C^∞, there exists a formal change of coordinates such that M is given by a formal surface $N(A)$, a surface which is defined by a formal power series of the form (3.1). In this case, (3.2) uniquely determines S as a formal power series. We sometimes identify a formal surface $N(A)$ in normal form with the collection of coefficients $A = (A^l_{\alpha\overline{\beta}})$ and denote by $\mathcal{N}$ the real vector space of all A satisfying (N1–3).

The conditions (N1–3) do not determine uniquely the normal form of a surface: two different surfaces in normal form may be (formally) biholomorphically equivalent. The equivalence classes of normal forms can be written as orbits in $\mathcal{N}$ of an action of the group of all fractional linear transformations which preserve the hyperquadric and the origin. To describe this action, let us first delineate the group explicitly.

In projective coordinates $(\zeta^0,\ldots,\zeta^n) \in \mathbb{C}^{n+1}$ defined by $z^j = \zeta^j/\zeta^0$, the hyperquadric is given by $\zeta^0\overline{\zeta}^n + \zeta^n\overline{\zeta}^0 - |\zeta'|^2 = 0$. Let g_0 denote the matrix

$$g_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -I_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{3.3}$$

Then, the hyperquadric is preserved by the induced linear fractional transformation ϕ_h of $\mathbb{C}^n$ for $h \in \mathrm{SU}(g_0)$. Clearly, ϕ_h leaves the origin $0 \in \mathbb{C}^n$ fixed if and only if h is in the subgroup

$$H = \{h \in \mathrm{SU}(g_0) : \ h\, e_0 = \lambda\, e_0, \ \lambda \in \mathbb{C}^*\},$$

where e_0 is the column vector ${}^t(1, 0, \dots, 0)$.

For $(h, A) \in H \times \mathcal{N}$, the surface $\phi_h(N(A))$ always fits the hyperquadric to the second order. Hence there is a unique transformation S normalized by (3.2) such that S sends $\phi_h(N(A))$ to a surface $N(\widetilde{A}) = S \circ \phi_h(N(A))$ in normal form. We set $\Phi_{(h,A)} = S \circ \phi_h$ and $h.A = \widetilde{A}$. Then the uniqueness of S implies

$$\Phi_{(\widetilde{h}h,A)} = \Phi_{(\widetilde{h},h.A)} \circ \Phi_{(h,A)} \quad \text{for any} \quad h, \widetilde{h} \in H. \tag{3.4}$$

Thus $H \times \mathcal{N} \ni (h, A) \mapsto h.A \in \mathcal{N}$ defines a left action of H on $\mathcal{N}$. Moreover, any biholomorphic map Φ such that $\Phi(0) = 0$ and $\Phi(N(A)) = N(\widetilde{A})$ is written as $\Phi = \Phi_{(h,A)}$ for some h satisfying $h.A = \widetilde{A}$. Therefore, the orbits of this H-action are the equivalence classes of the normal form.

We now rewrite the definition of CR invariants in terms of this H-action.

Definition 3.1. An *H-invariant of $\mathcal{N}$ of weight w* is a polynomial $I(A)$ in the components of $A = (A^l_{\alpha\overline{\beta}})$ satisfying

$$h.I = \sigma_{w,w}(h) I \quad \text{for } h \in H, \tag{3.5}$$

where $(h.I)(A) = I(h^{-1}.A)$ and $\sigma_{p,q}$ is the character of H given by

$$\sigma_{p,q}(h) = \lambda^{-p}\overline{\lambda}^{-q} \quad \text{when} \quad h\, e_0 = \lambda\, e_0.$$

We denote by $I^w(\mathcal{N})$ the vector space of all invariants of $\mathcal{N}$ of weight w.

Let us observe that (3.5) is equivalent to (0.2) and hence $I(A)$ is an H-invariant if and only if it is a CR invariant. If $\Phi(N(A)) = N(\widetilde{A})$ and $\Phi(0) = 0$, then there exists an $h \in H$ such that $\widetilde{A} = h.A$ and $\Phi = \Phi_{(h,A)}$. Since $\Phi'(0) = \phi'_h(0)$ and $\det \phi'_h(0) = \lambda^{-n-1}$,

$$|\det \Phi'(0)|^{-2w/(n+1)} = |\det \phi'_h(0)|^{-2w/(n+1)} = \sigma_{-w,-w}(h). \tag{3.6}$$

Thus (0.2) is written as $I(h.A) = \sigma_{-w,-w}(h) I(A)$, which is equivalent to (3.5).

3.2. *Action of H on the defining functions.* In the previous subsection, we expressed the transformation law of CR invariants in terms of the H-action on $\mathcal{N}$. Proceeding similarly, let us express the transformation law of defining functions (1.6) by using the group H.

We consider asymptotic solutions to (1.2) for a surface $N(A)$ in normal form, where r and η_k are regarded as real formal power series about $0 \in \mathbb{C}^n$. Let $\mathcal{F}_{N(A)}$ denote the totality of the smooth parts of asymptotic solutions for

$N(A)$. Then, applying the argument in the proof of Proposition 1, we can find for any real formal power series

$$h_C(z', \overline{z}', v) = \sum_{|\alpha|, |\beta|, l \geq 0} C^l_{\alpha\overline{\beta}} z'^\alpha \overline{z}'^\beta v^l, \tag{3.7}$$

a unique formal power series $r \in \mathcal{F}_{N(A)}$ such that

$$\partial_\rho^{n+2} r|_{\rho=0} = h_C, \tag{3.8}$$

where ∂_ρ is the differentiation with respect to ρ in the (formal) coordinates $(z', \overline{z}', v, \rho)$ given by (3.1). Thus, denoting by $\mathcal{C}$ the totality of the collections of coefficients $C = (C^l_{\alpha\overline{\beta}})$ of (3.7), we can define a map

$$\iota_1 : \mathcal{N} \times \mathcal{C} \to \mathcal{F} = \bigcup_{A \in \mathcal{N}} \mathcal{F}_{N(A)}$$

which assigns to each $(A, C) \in \mathcal{N} \times \mathcal{C}$ the element $r \in \mathcal{F}_{N(A)}$ satisfying (3.8).

For $h \in H$ and $r \in \mathcal{F}_{N(A)}$, we define $h.r = \widetilde{r} \in \mathcal{F}_{N(h.A)}$ by

$$\widetilde{r} \circ \Phi_{(h,A)} = |\det \Phi'_{(h,A)}|^{2/(n+1)} r.$$

Then the map $H \times \mathcal{F} \ni (h, r) \mapsto h.r \in \mathcal{F}$ gives an H-action on $\mathcal{F}$ in virtue of (3.4), and it induces an H-action on $\mathcal{N} \times \mathcal{C}$ via the bijection ι_1. With respect to this H-action, we can characterize invariants of the pair (M, r) as H-invariant polynomials of $\mathcal{N} \times \mathcal{C}$ defined as follows.

Definition 3.2. An *H-invariant of $\mathcal{N} \times \mathcal{C}$ of weight w* is a polynomial $I(A, C)$ in the variables $A^l_{\alpha\overline{\beta}}, C^l_{\alpha\overline{\beta}}$ satisfying (3.5) with

$$(h.I)(A, C) = I(h^{-1}.(A, C)).$$

The *totality* of such polynomials is denoted by $I^w(\mathcal{N} \times \mathcal{C})$.

Observe that the projection $\mathcal{N} \times \mathcal{C} \to \mathcal{N}$ is H-equivariant. Hence an H-invariant of $\mathcal{N}$ is regarded as an H-invariant of $\mathcal{N} \times \mathcal{C}$, and $I^w(\mathcal{N})$ is identified with a subspace of $I^w(\mathcal{N} \times \mathcal{C})$.

3.3. *Tensorial realization.* We next embed the H-space $\mathcal{F}$, and also $\mathcal{N} \times \mathcal{C}$, into a tensor H-module by using the curvatures of the ambient metrics $g[r]$. Recall that, for $r \in \mathcal{F}$, the ambient metric is defined by the Kähler potential $r_\#$, and $r_\#$ is a formal power series in z^0, z (and $\overline{z}^0, \overline{z}$) about $e_0 = (1, 0) \in \mathbb{C}^* \times \mathbb{C}^n$. Since $r_\#$ is homogeneous in z^0, it follows that the ambient metric, the curvature tensor $R_{j\overline{k}l\overline{m}}$ and the covariant derivatives $R_{j\overline{k}l\overline{m},\gamma_1\cdots\gamma_p}$ are defined at each point $\lambda e_0 \in \mathbb{C}^* \times \mathbb{C}^n$, $\lambda \in \mathbb{C}^*$.

For simplicity of the notation, we write $R^{(p,q)} = (R_{\alpha\overline{\beta}})_{|\alpha|=p, |\beta|=q}$, where

$$R_{\alpha\overline{\beta}} = \begin{cases} R_{\alpha_1\overline{\beta}_1\alpha_2\overline{\beta}_2,\alpha_3\cdots\alpha_p\overline{\beta}_3\cdots\overline{\beta}_q} & \text{if } |\alpha| \geq 2 \text{ and } |\beta| \geq 2; \\ 0 & \text{otherwise.} \end{cases}$$

Here, components of tensors are written with respect to the coordinates $\zeta = (\zeta^0, \zeta^1, \dots, \zeta^n) \in \mathbb{C}^{n+1}$ given by

$$\zeta^0 = z^0,\ \zeta^1 = z^0 z^1,\ \dots,\ \zeta^n = z^0 z^n.$$

Then we have, at the point e_0,

$$\begin{aligned} R_{\alpha' 0 \alpha'' \overline{\beta}} &= (1 - |\alpha'\alpha''|) R_{\alpha'\alpha''\overline{\beta}}, \\ R_{\alpha \overline{\beta'}\,\overline{0}\,\overline{\beta''}} &= (1 - |\beta'\beta''|) R_{\alpha \overline{\beta'}\,\overline{\beta''}}, \end{aligned} \tag{3.9}$$

for all lists $\alpha, \alpha', \alpha'', \beta, \beta', \beta''$ of indices in $\{0, 1, \dots, n\}$ with $|\alpha'|, |\beta'| > 1$. This fact is a consequence of the homogeneity of the Kähler potential $r_\#(\zeta, \overline{\zeta})$ in ζ and $\overline{\zeta}$; see the tensor restriction lemma in [F3].

We write down the transformation law of $R_{\alpha\overline{\beta}}$ which comes from the H-action on $\mathcal{F}$. Let $W = \mathbb{C}^{n+1}$ denote the defining representation of $\mathrm{SU}(g_0)$, hence also of H, by left multiplication on the column vectors. We define H-modules

$$\mathbf{T}_s^{p,q} = (\otimes^{p,q} W^*) \otimes \sigma_{s-p,s-q}, \quad \text{for } p, q, s \in \mathbb{Z} \text{ with } p, q \geq 0,$$

where $\otimes^{p,q} W^* = (\otimes^p W^*) \otimes (\otimes^q \overline{W^*})$. We denote by $\mathbf{T}_s$ the H-submodule of $\prod_{p,q\geq 0} \mathbf{T}_s^{p,q}$ consisting of all $T = (T_{\alpha\overline{\beta}})_{|\alpha|,|\beta|\geq 0}$ satisfying

$$\begin{aligned} T_{\alpha' 0 \alpha'' \overline{\beta}} &= (s - |\alpha'\alpha''|) T_{\alpha'\alpha''\overline{\beta}}, \\ T_{\alpha \overline{\beta'}\,\overline{0}\,\overline{\beta''}} &= (s - |\beta'\beta''|) T_{\alpha \overline{\beta'}\,\overline{\beta''}}, \end{aligned} \tag{3.10$_s$}$$

where $\alpha, \alpha', \alpha'', \beta, \beta', \beta''$ are lists of indices with $|\alpha'|, |\beta'| > s$. Then, (3.9) permits us to define a map $\iota_2 \colon \mathcal{F} \to \mathbf{T}_1$ by setting $\iota_2(r) = (R_{\alpha\overline{\beta}}|_{e_0})_{|\alpha|,|\beta|\geq 0}$, where $R_{\alpha\overline{\beta}}$ are the components of the covariant derivatives of the curvature of $g[r]$.

Proposition 3.3. *The map ι_2 is H-equivariant. In particular, the image $\mathcal{R} = \iota_2(\mathcal{F})$ is an H-invariant subset of $\mathbf{T}_1$.*

Proof. For $r \in \mathcal{F}_{N(A)}$ and $h \in H$, set $\widetilde{r} = h.r$. Then $g[\widetilde{r}] = F_* g[r]$, where $F = (\Phi_{(h,A)})_\#$, so that

$$\widetilde{R}^{(p,q)} = F_* R^{(p,q)} \quad \text{for any } p, q \geq 0,$$

where $R^{(p,q)}$ and $\widetilde{R}^{(p,q)}$ are curvatures of $g[r]$ and $g[\widetilde{r}]$, respectively. Evaluating this formula at e_0, we have

$$\iota_2(h.r) = \Big((F_* R^{(p,q)})|_{e_0} \Big)_{p,q\geq 0}.$$

Note that the right-hand side is independent of the choice of the lift F. We shall fix F as in the next lemma and express $(F_* R^{(p,q)})|_{e_0}$ in terms of $R^{(p,q)}|_{e_0}$ and h.

LEMMA 3.4. *There exists a lift F of $\Phi_{(h,A)}$ satisfying $F(e_0) = \lambda\, e_0$, where $\lambda = \sigma_{-1,0}(h)$. For such a lift F, the Jacobian matrix F' at $\lambda^{-1}e_0$ with respect to ζ is h.*

Proof of Lemma 3.4. We first note that the linear map $\zeta \mapsto h\,\zeta$ gives a lift of ϕ_h. This lift satisfies $(\phi_h)_\#(e_0) = \lambda\, e_0$ and $(\phi_h)'_\#(\nu\, e_0) = h$ for any $\nu \in \mathbb{C}^*$. On the other hand, for a map S normalized by (3.2), we can define $S_\#$ such that $S_\#(e_0) = e_0$ and $S'_\#(e_0) = \mathrm{id}$; see Lemma N1 of [F3]. Hence the composition $F = S_\# \circ (\phi_h)_\#$ gives a lift of $\Phi_{(h,A)} = S \circ \phi_h$ satisfying $F(e_0) = \lambda\, e_0$. The Jacobian matrix of F at $\lambda^{-1}e_0$ is given by $F'(\lambda^{-1}e_0) = S'_\#(e_0) \cdot (\phi_h)'_\#(\lambda^{-1}e_0) = h$. $\square$

By this lemma, we see that $(F_* R^{(p,q)})|_{e_0}$ is given by the usual tensorial action of h on $R^{(p,q)}|_{\lambda^{-1}e_0} \in \otimes^{p,q} W^*$. To compare $R^{(p,q)}|_{\lambda^{-1}e_0}$ with $R^{(p,q)}|_{e_0}$, we next consider the scaling map $\phi(\zeta) = \lambda\zeta$. Then, from the homogeneity of g, we have $\phi_* g = |\lambda|^{-2} g$, so that $\phi_* R^{(p,q)} = |\lambda|^{-2} R^{(p,q)}$. Thus we get $R^{(p,q)}|_{\lambda^{-1}e_0} = \lambda^{p-1}\overline{\lambda}^{q-1} R^{(p,q)}|_{e_0}$. Summing up, we obtain $(F_* R^{(p,q)})|_{e_0} = h.(R^{(p,q)}|_{e_0}) \in \mathbf{T}_1^{p,q}$. $\square$

We have defined the following H-equivariant maps:

$$\mathcal{N} \times \mathcal{C} \xrightarrow{\iota_1} \mathcal{F} \xrightarrow{\iota_2} \mathbf{T}_1.$$

We set $\iota = \iota_2 \circ \iota_1$. Inspecting the construction of ι_1 and ι_2, we see that ι is a polynomial map in the sense that each component of $\iota(A, C) = (T_{\alpha\overline{\beta}}(A, C))$ is a polynomial in the variables $(A^l_{\alpha\overline{\beta}}, C^l_{\alpha\overline{\beta}})$.

We now define H-invariants of $\mathcal{R} = \iota(\mathcal{N} \times \mathcal{C}) \subset \mathbf{T}_1$.

Definition 3.5. An *H-invariant of $\mathcal{R}$ of weight w* is a polynomial $I(T)$ in the components of $(T_{\alpha\overline{\beta}}) \in \mathbf{T}_1$ satisfying

$$I(h^{-1}.T) = \sigma_{w,w}(h) I(T) \quad \text{for any } (h, T) \in H \times \mathcal{R}.$$

Identifying two H-invariants which agree on $\mathcal{R}$, we denote by $I^w(\mathcal{R})$ the totality of the equivalence classes of all H-invariants of $\mathcal{R}$ of weight w.

This definition implies that ι induces an injection

$$\iota^* \colon I^w(\mathcal{R}) \ni I(T) \mapsto I(\iota(A, C)) \in I^w(\mathcal{N} \times \mathcal{C}). \tag{3.11}$$

This map is also surjective by the following theorem.

THEOREM 4. *There exists a polynomial map $\tau \colon \mathbf{T}_1 \to \mathcal{N} \times \mathcal{C}$ such that $\tau \circ \iota = \mathrm{id}$. In particular, ι is injective and thus the map* (3.11) *is an isomorphism.*

On the tensor space $\mathbf{T}_1$, we can construct H-invariants by making complete contractions of the form

$$\mathrm{contr}\left(T^{(p_1,q_1)} \otimes \cdots \otimes T^{(p_d,q_d)}\right),$$

where $T^{(p,q)} \in \mathbf{T}_1^{p,q}$ and the contraction is taken with respect to the metric g_0. We call such H-invariants *elementary invariants*. The next theorem asserts that $I^w(\mathcal{R})$ is spanned by the elementary invariants of weight w.

THEOREM 5. *Every H-invariant of $\mathcal{R}$ coincides on $\mathcal{R}$ with a linear combination of elementary invariants of* $\mathbf{T}_1$.

Once we know Theorems 4 and 5, we can easily prove the main theorems stated in Section 1.

3.4. *Proofs of the main theorems using Theorems* 4 *and* 5.

Proof of Theorem 1. We here consider only the log term $\psi[r]$. The expansion of $\varphi[r]$ is obtained exactly in the same manner if we note that $\varphi[r]$ is defined only up to $O^{n+1}(\partial\Omega)$ and hence one should keep track of the ambiguity in each step of the proof; see [F3].

We prove by induction on m that there exist Weyl functionals W_k of weight k such that

$$(3.12)_m \qquad \psi[r] = \sum_{k=0}^{m-1} W_{k+n+1}[r]\, r^k + J_m[r]\, r^m \quad \text{for} \quad J_m[r] \in C^\infty(\overline{\Omega}).$$

We interpret $(3.12)_0$ as $\psi[r] \in C^\infty(\overline{\Omega})$. Assuming $(3.12)_m$, we seek a Weyl functional W_{m+n+1} such that $J_m[r] = W_{m+n+1}[r]$ on $\partial\Omega$ for any Ω and $r \in \mathcal{F}_{\partial\Omega}$. Then, $(3.12)_{m+1}$ is obtained by setting $J_{m+1}[r] = (J_m[r] - W_{m+n+1}[r])/r$.

We first study the transformation law of J_m under a biholomorphic map $\Phi\colon \Omega \to \widetilde{\Omega}$. Let $\widetilde{K}$ be the Bergman kernel of $\widetilde{\Omega}$ and $\psi[\widetilde{r}]$ its log term. It then follows from $\widetilde{K} \circ \Phi = |\det \Phi'|^{-2} K$ that

$$(3.13) \qquad \psi[\widetilde{r}] \circ \Phi = |\det \Phi'|^{-2} \psi[r] \mod O^\infty(\partial\Omega).$$

Choosing $\widetilde{r}$ as in Proposition 2, (1.6), we obtain

$$(3.14) \qquad J_m[\widetilde{r}] \circ \Phi = |\det \Phi'|^{-2(m+n+1)/(n+1)} J_m[r] \mod O^\infty(\partial\Omega).$$

That is, J_m satisfies a transformation law of weight $m+n+1$. If $\partial\Omega$ and $\partial\widetilde{\Omega}$ are locally written in normal form $N(h^{-1}.A)$ and $N(A)$, respectively, then the restriction of (3.14) to $z = 0$ gives

$$J_m[\widetilde{r}](0) = \sigma_{m+n+1,m+n+1}(h) J_m[r](0).$$

From this formula, we can conclude $J_m[r](0) \in I^{m+n+1}(\mathcal{N} \times \mathcal{C})$ if we know that $J_m[r](0)$ is a polynomial in the components of $(A, C) \in \mathcal{N} \times \mathcal{C}$. Now we have:

LEMMA 3.6. *Assume that $\partial\Omega$ is locally written in normal form $N(A)$, $A \in \mathcal{N}$ and that $r = \iota_1(A, C)$. Then $J_m[r](0)$ is a universal polynomial in the components of (A, C).*

Proof. Inspecting the proof of the existence of asymptotic expansion (1.9) in [F1] or [BS], we see that the Taylor coefficients of ψ about 0 are universal polynomials in A (direct methods of computing these universal polynomials are given in [B1,2] and [HKN1,2]). On the other hand, by the constructions of r and Weyl invariants, we can show that the Taylor coefficients of r and of $W_k[r]$ about 0 are universal polynomials in (A, C). Therefore, we see by the relation $(3.12)_m$ that $J_m[r](0)$ is a universal polynomial in (A, C). $\square$

We now apply Theorems 3 and 4 to obtain a Weyl functional W_{m+n+1} such that $J_m[r](0) = W_{m+n+1}[r](0)$ for any $\partial\Omega$ in normal form and any $r \in \mathcal{F}_{\partial\Omega}$. Noting that J_m and W_{m+n+1} satisfy the same transformation law under biholomorphic maps, we conclude $J_m[r] = W_{m+n+1}[r]$ on $\partial\Omega$ for any Ω by locally transforming $\partial\Omega$ into normal form. We thus complete the inductive step. $\square$

Proof of Theorem 2. Note that the isomorphism $\iota^*\colon I^w(\mathcal{R}) \to I^w(\mathcal{N} \times \mathcal{C})$ of Theorem 4 is given by $W(R) \mapsto I_W(A, C)$. Thus Theorem 5 guarantees that each element of $I^w(\mathcal{N} \times \mathcal{C})$ is expressed as a Weyl invariant $I_W(A, C)$ of weight w. Noting that $I_W \in I^w(\mathcal{N})$ if and only if I_W is $\mathcal{C}$-independent, we obtain Theorem 2. $\square$

4. Proof of Theorem 4

4.1. *Reduction to finite-dimensional cases.* We first write the map ι as a projective limit of maps ι_m, $m = 1, 2, \ldots$, on finite-dimensional H-spaces. Then, we can reduce the proof of Theorem 4 to that of an analogous assertion for each ι_m (Proposition 4.1 below).

To define the maps ι_m, we introduce weights for the components of $\mathcal{N}$, $\mathcal{C}$ and $\mathbf{T}_s$ by considering the action of the matrix

$$\delta_t = \begin{pmatrix} t & 0 & 0 \\ 0 & I_{n-1} & 0 \\ 0 & 0 & 1/t \end{pmatrix} \in H, \quad t > 0.$$

The actions of δ_t on $(A^l_{\alpha\overline{\beta}}, C^l_{\alpha\overline{\beta}}) \in \mathcal{N} \times \mathcal{C}$ and $(T_{\alpha\overline{\beta}}) \in \mathbf{T}_s$ are given by

$$\begin{aligned} \delta_t.(A^l_{\alpha\overline{\beta}}, C^l_{\alpha\overline{\beta}}) &= (t^{|\alpha|+|\beta|+2l-2} A^l_{\alpha\overline{\beta}}, t^{|\alpha|+|\beta|+2l+2(n+1)} C^l_{\alpha\overline{\beta}}), \\ \delta_t.(T_{\alpha\overline{\beta}}) &= (t^{\|\alpha\overline{\beta}\|-2s} T_{\alpha\overline{\beta}}). \end{aligned}$$

Here $\|\alpha\overline{\beta}\|$ is the strength of the indices defined by setting $\|a_1 \ldots a_k\| = \|a_1\| + \cdots + \|a_k\|$ with $\|0\| = \|\overline{0}\| = 0$, $\|j\| = \|\overline{\jmath}\| = 1$ for $1 \le j \le n-1$ and

$\|n\| = \|\overline{n}\| = 2$. The *weights* of the components $A^l_{\alpha\overline{\beta}}$, $C^l_{\alpha\overline{\beta}}$ and $T_{\alpha\overline{\beta}}$ are defined to be the halves of degrees in t with respect to the action of δ_t:

$$\frac{1}{2}(|\alpha| + |\beta|) + l - 1, \quad \frac{1}{2}(|\alpha| + |\beta|) + l + n + 1 \quad \text{and} \quad \frac{1}{2}\|\alpha\overline{\beta}\| - s,$$

respectively. We also define the weight for a monomial of $(A^l_{\alpha\overline{\beta}}, C^l_{\alpha\overline{\beta}})$ or $(T_{\alpha\overline{\beta}})$ to be the half of the degree with respect to δ_t. Then the notion of weight is consistent with that for H-invariants.

Let $[\mathcal{N}]_m$ denote the vector space of all components $A^l_{\alpha\overline{\beta}}$ of weight $\leq m$:

$$[\mathcal{N}]_m = \left\{ (A^l_{\alpha\overline{\beta}})_{(|\alpha|+|\beta|)/2+l-1\leq m} : (A^l_{\alpha\overline{\beta}}) \in \mathcal{N} \right\}.$$

We also define $[\mathcal{C}]_m$ and $[\mathbf{T}_s]_m$ similarly. Then $[\mathcal{N}]_m$, $[\mathcal{C}]_m$ and $[\mathbf{T}_s]_m$ are finite-dimensional vector spaces such that their projective limits as $m \to \infty$ are $\mathcal{N}$, $\mathcal{C}$ and $\mathbf{T}_s$, respectively.

Since ι is compatible with the action of δ_t, each component $T_{\alpha\overline{\beta}}(A, C)$ of $\iota(A, C)$ is a homogeneous polynomial of weight $||\alpha\overline{\beta}||/2 - 1$. It follows that, if $||\alpha\overline{\beta}||/2 - 1 \leq m$, then $T_{\alpha\overline{\beta}}(A, C)$ depends only on the variables $(A, C) \in [\mathcal{N}]_m \times [\mathcal{C}]_m$, because all components of $(A, C) \in \mathcal{N} \times \mathcal{C}$ have positive weight. We can thus define maps

$$\iota_m : [\mathcal{N}]_m \times [\mathcal{C}]_m \to [\mathbf{T}_1]_m \quad \text{by} \quad \iota_m(A, C) = (T_{\alpha\overline{\beta}}(A, C))_{||\alpha\overline{\beta}||/2-1\leq m}.$$

The projective limit of ι_m as $m \to \infty$ gives ι. Therefore, Theorem 4 is reduced to the following finite-dimensional proposition.

PROPOSITION 4.1. *There exist polynomial maps* $\tau_m : [\mathbf{T}_1]_m \to [\mathcal{N}]_m \times [\mathcal{C}]_m$, *for* $m = 0, 1, 2 \ldots$, *such that* $\tau_m \circ \iota_m = \mathrm{id}$ *and* $\pi_m \circ \tau_m = \tau_{m-1} \circ \pi'_m$. *Here* $\pi_m : [\mathcal{N}]_m \times [\mathcal{C}]_m \to [\mathcal{N}]_{m-1} \times [\mathcal{C}]_{m-1}$ *and* $\pi'_m : [\mathbf{T}_1]_m \to [\mathbf{T}_1]_{m-1}$ *are the natural surjections.*

The projective limit of τ_m is a polynomial map $\tau : \mathbf{T}_1 \to \mathcal{N} \times \mathcal{C}$ such that $\tau \circ \iota = \mathrm{id}$. Thus Theorem 4 follows from Proposition 4.1.

4.2. *Linearization of the Monge-Ampère equation.* We prove Proposition 4.1 by using the inverse function theorem. Our first task is to show the injectivity of $d\iota_m|_0 : T_0([\mathcal{N}]_m \times [\mathcal{C}]_m) \to [\mathbf{T}_1]_m$, the differential of ι_m at $0 = (0, 0)$. Identifying $T_0([\mathcal{N}]_m \times [\mathcal{C}]_m)$ with $[\mathcal{N}]_m \oplus [\mathcal{C}]_m$, we define a linear map $d\iota|_0 : \mathcal{N} \oplus \mathcal{C} \to \mathbf{T}_1$ by the projective limit of $d\iota_m|_0$. Then we can prove the injectivity of each $d\iota_m|_0$ by proving that of $d\iota|_0$.

Before starting the proof, we introduce some vector spaces of formal power series, which will be used in the rest of this paper. For $s \in \mathbb{Z}$, let $\mathcal{E}(s)$ denote the vector space of real formal power series $f(\zeta, \overline{\zeta})$ of $\zeta, \overline{\zeta}$ about $e_0 \in \mathbb{C}^{n+1}$ of homogeneous degree (s, s). Here we say that f is homogeneous of degree (s, s') if $Zf = s\, f$ and $\overline{Z}f = s'\, f$, where $Z = \sum_{j=0}^{n} \zeta^j \partial_{\zeta^j}$. The space $\mathcal{E}(s)$

admits a natural H-action $(h.f)(\zeta,\overline{\zeta}) = f(h^{-1}\zeta, \overline{h^{-1}\zeta})$, and is embedded as an H-submodule of $\mathbf{T}_s$ by $f \mapsto (T_{\alpha\overline{\beta}})_{|\alpha|,|\beta|\geq 0}$, where $T_{\alpha\overline{\beta}} = \partial_\zeta^\alpha \partial_{\overline{\zeta}}^\beta f(e_0)$. Using this expression, we define H-submodules of $\mathcal{E}(s)$ for $s \geq 0$:

$$\begin{aligned}\mathcal{E}_s &= \{f \in \mathcal{E}(s) : T_{\alpha\overline{\beta}} = 0 \text{ if } |\alpha| \leq s \ \text{ or } \ |\beta| \leq s\},\\ \mathcal{E}^s &= \{f \in \mathcal{E}(s) : T_{\alpha\overline{\beta}} = 0 \text{ if } |\alpha| > s \text{ and } |\beta| > s\}.\end{aligned}$$

Then we have a decomposition of $\mathcal{E}(s)$ as H-modules

$$\mathcal{E}(s) = \mathcal{E}_s \oplus \mathcal{E}^s. \tag{4.1}$$

We next consider the restrictions of $f \in \mathcal{E}(s)$ to the null cone

$$\mathcal{Q} = \{\mu = \zeta^0\overline{\zeta}^n + \zeta^n\overline{\zeta}^0 - |\zeta'|^2 = 0\} \subset \mathbb{C}^{n+1}$$

associated with g_0 and set

$$\mathcal{J}(s) = \{f|_\mathcal{Q} : f \in \mathcal{E}(s)\}, \quad \mathcal{J}^s = \{f|_\mathcal{Q} : f \in \mathcal{E}^s\}. \tag{4.2}$$

If we employ $(z^0, \overline{z}^0, z', \overline{z}', v)$ as coordinates of $\mathcal{Q}$, then each $f \in \mathcal{J}(s)$ is written as

$$f = |z^0|^{2s} \sum_{|\alpha|,|\beta|,l\geq 0} B^l_{\alpha\overline{\beta}} z'^\alpha \overline{z}'^\beta v^l. \tag{4.3}$$

Thus we may identify $\mathcal{J}(s)$ with the space of real formal power series in $(z', \overline{z}', v)$. Using this identification, we embed $\mathcal{N}$ as a subspace of $\mathcal{J}(1)$ and identify $\mathcal{C}$ with $\mathcal{J}(-n-1)$, so that the actions of δ_t on $\mathcal{N}$ and $\mathcal{C}$ are compatible with those on $\mathcal{J}(1)$ and $\mathcal{J}(-n-1)$, respectively. It then follows from [CM] that

$$\mathcal{J}(1) = \mathcal{N} \oplus \mathcal{J}^1 \quad \text{(direct sum of vector spaces)}. \tag{4.4}$$

This is the key equation in the proof of the uniqueness of the normal form.

Remark 4.2. (i) In the decomposition (4.4), the vector space $\mathcal{J}^1$, as well as $\mathcal{N}$, is realized by a subspace of $\mathcal{J}(1)$. In [CM], elements of $\mathcal{J}(1)$ are expressed by formal power series of $(z', \overline{z}', v)$, where $\mathcal{J}^1$ is identified via (4.3) with the range of a linear operator L defined by

$$L(F) = \mathrm{Re}\left(\overline{z}^1 f_1 + \cdots + \overline{z}^{n-1} f_{n-1} + f_n\right)\Big|_{u=|z'|^2/2}$$

for $\mathbb{C}^n$-valued formal power series $F = (f_1, \ldots, f_n)$ of z.

(ii) The H-action on $\mathcal{J}(1)$ is linear, whereas that on $\mathcal{N}$ is nonlinear. These are defined differently, and $\mathcal{N}$ in (4.4) is not an H-invariant subspace of $\mathcal{J}(1)$. Nevertheless, the tangent space $T_0\mathcal{N}$ at the origin is H-isomorphic to $\mathcal{J}(1)/\mathcal{J}^1$ as follows. Let us tentatively introduce an H-submodule $\mathcal{J}_3(1)$ of $\mathcal{J}(1)$ representing surfaces close to those in normal form; $\mathcal{J}_3(1)$ consists of elements of

$\mathcal{J}(1)$ vanishing to the third order at e_0, and each $f \in \mathcal{J}_3(1)$ is identified with the surface $N(f) = N(B)$ via (4.3). Then the H-action $(h, N(f)) \mapsto \phi_h(N(f))$ on $\mathcal{J}_3(1)$ is nonlinear, while the linearization coincides with the action $\rho(h)$ on $\mathcal{J}(1)$. Now $\mathcal{N}$ is a subset of $\mathcal{J}_3(1)$, and the linearization of the H-action on $\mathcal{N}$ is given by $(h, f) \mapsto p(\rho(h)f)$, where $p\colon \mathcal{J}(1) \to \mathcal{N}$ denotes the projection associated with the decomposition (4.4). This is the H-action on $T_0\mathcal{N}$, so that $T_0\mathcal{N}$ is H-isomorphic to the quotient space $\mathcal{J}(1)/\mathcal{J}^1$.

(iii) In [CM], surfaces in normal form are constructed within $\mathcal{J}_3(1)$ by induction on weight. It should be noted that the grading by weight is different from the linearization. More precisely, the lowest weight part in the deviation from normal form is determined by the projection $p\colon \mathcal{J}(1) \to \mathcal{N}$, whereas higher weight parts are affected nonlinearly by lower weight terms. We use a similar procedure in the remaining part of this section, where the induction is implicit in the inverse function theorem.

Using these power series, we shall write down $d\iota|_0(A, C)$ explicitly. By definition,

$$d\iota|_0(A, C) = \frac{d}{d\varepsilon}\iota(\varepsilon\, A, \varepsilon\, C)\Big|_{\varepsilon=0}.$$

To compute the right-hand side, we consider a family of asymptotic solutions

$$U_\varepsilon = r_{\varepsilon\#} + r_{\varepsilon\#}\sum_{k=1}^{\infty}\eta_{\varepsilon,k}\left(r_\varepsilon^{n+1}\log r_{\varepsilon\#}\right)^k \tag{4.5}$$

with $r_\varepsilon = \iota_1(\varepsilon\, A, \varepsilon\, C)$, and derive an equation such that $(dU_\varepsilon/d\varepsilon)|_{\varepsilon=0}$ is a unique solution.

PROPOSITION 4.3. (i) $F = (dU_\varepsilon/d\varepsilon)|_{\varepsilon=0}$ *admits an expansion of the form*

$$F = \varphi + \eta\, \mu^{n+2}\log\mu, \quad \textit{where } \varphi \in \mathcal{E}(1),\ \eta \in \mathcal{E}(-n-1). \tag{4.6}$$

(ii) *Let ∂_μ denote the differentiation with respect to μ for the coordinates $(z^0, \overline{z}^0, z', \overline{z}', v, \mu)$. Then*

$$\varphi|_{\mathcal{Q}} = -f_A, \quad \textit{where} \quad f_A = |z^0|^2\sum A^l_{\alpha\overline{\beta}} z'^\alpha \overline{z}'^\beta v^l, \tag{4.7}$$

$$\partial_\mu^{n+2}\varphi|_{\mathcal{Q}} = h_C, \quad \textit{where} \quad h_C = |z^0|^{-2n-2}\sum C^l_{\alpha\overline{\beta}} z'^\alpha \overline{z}'^\beta v^l. \tag{4.8}$$

(iii) *Let*

$$\Delta = \partial_{\zeta^0}\partial_{\overline{\zeta}^n} + \partial_{\zeta^n}\partial_{\overline{\zeta}^0} - \sum_{j=1}^{n-1}\partial_{\zeta^j}\partial_{\overline{\zeta}^j}$$

be the Laplacian for the ambient metric g_0 with potential μ. Then $\Delta F = 0$.

(iv) *For each $(f, h) \in \mathcal{J}(1) \oplus \mathcal{J}(-n-1)$, there exists a unique function F of the form* (4.6) *satisfying $\Delta F = 0$ and*

$$\varphi|_{\mathcal{Q}} = -f, \quad \partial_\mu^{n+2}\varphi|_{\mathcal{Q}} = h. \tag{4.9}$$

In the proof of (i), we use the following lemma, which implies, in particular, that the first variation of the higher log terms in (4.5) vanishes at $\varepsilon = 0$.

LEMMA 4.4. *Each log term coefficient of U_ε satisfies $\eta_{\varepsilon,k} = O(\varepsilon^k)$.*

Proof. Setting $U_\varepsilon = r_{\varepsilon\#}(1 + f_\varepsilon)$, we shall show that $f_\varepsilon \in O_{\mathcal{A}}(\varepsilon)$, where $O_{\mathcal{A}}(\varepsilon)$ is the space of all formal series of the form

$$\sum_{k=1}^{\infty} \eta_{\varepsilon,k} \left(r_\varepsilon^{n+1} \log r_{\varepsilon\#}\right)^k \quad \text{with} \quad \eta_{\varepsilon,k} = O(\varepsilon^k).$$

Neglecting higher log terms in f_ε, we set $\widetilde{f}_\varepsilon = \eta_{\varepsilon,1}\, r_\varepsilon^{n+1} \log r_{\varepsilon\#}$. Then we have $\widetilde{f}_\varepsilon \in O_{\mathcal{A}}(\varepsilon)$ and $\mathcal{M}_\varepsilon(\widetilde{f}_\varepsilon) = J(r_\varepsilon)^{-1} \mod \mathcal{A}^\varepsilon_{2(n+1)}$, where $\mathcal{M}_\varepsilon$ and $\mathcal{A}^\varepsilon_{2(n+1)}$ are defined with respect to $\rho = r_\varepsilon$. We can recover f_ε from $\widetilde{f}_\varepsilon$ by the procedure of constructing asymptotic solutions used in the proof of Lemma 2.2. This procedure consists of the operations of applying $\mathcal{M}_\varepsilon$ and solving the equation $E_\varepsilon[g_\varepsilon]_{\underline{s}} = [h_\varepsilon]_s$ for $s > n + 1$. These operations preserve the class $O_{\mathcal{A}}(\varepsilon)$, so that $\widetilde{f}_\varepsilon \in O_{\mathcal{A}}(\varepsilon)$ implies $f_\varepsilon \in O_{\mathcal{A}}(\varepsilon)$ as desired. □

Proof of Proposition 4.3. (i) By Lemma 4.4 above, we have

$$U_\varepsilon = r_{\varepsilon\#} + \mu^{n+2}|z^0|^{-2(n+1)}\eta_{\varepsilon,1} \log \mu + O(\varepsilon^2).$$

We thus get (4.6) with

$$\varphi = |z^0|^2 (d\, r_\varepsilon/d\varepsilon)|_{\varepsilon=0} \quad \text{and} \quad \eta = |z^0|^{-2(n+1)} (d\, \eta_{\varepsilon,1}/d\varepsilon)|_{\varepsilon=0}.$$

(ii) We now regard f_A as an element of $\mathcal{E}(1)$ which is independent of μ in the coordinates $(z^0, \overline{z}^0, z', \overline{z}', v, \mu)$. Setting $\mu_\varepsilon = \mu - \varepsilon\, f_A$, we then consider the Taylor expansion of $r_{\varepsilon\#}$ with respect to μ_ε in the coordinates $(z^0, \overline{z}^0, z', \overline{z}', v, \mu_\varepsilon)$:

$$r_{\varepsilon\#} = \mu_\varepsilon + \sum_{k=1}^{\infty} a_{\varepsilon,k}(z^0, \overline{z}^0, z', \overline{z}', v)\, \mu_\varepsilon^k. \tag{4.10}$$

Since $r_{\varepsilon\#} = \mu + O(\varepsilon)$, we have $a_{\varepsilon,k} = O(\varepsilon)$. Differentiating both sides of (4.10) with respect to ε and then setting $\varepsilon = 0$, we get

$$\varphi = -f_A + \sum_{k=1}^{\infty} a'_k\, \mu^k, \quad \text{where} \quad a'_k = \left.\frac{d\, a_{\varepsilon,k}}{d\, \varepsilon}\right|_{\varepsilon=0}. \tag{4.11}$$

Restricting this formula to $\mu = 0$, we obtain (4.7). We also have by (4.11) that $\partial_\mu^{n+2}\varphi|_{\mu=0} = (n+2)!\, a'_{n+2}$. Therefore, noting that (3.8) is equivalent to $a_{\varepsilon,n+2} = \varepsilon\, h_C/(n+2)!$, we obtain (4.8).

(iii) Recalling that g_0 is the flat metric given by the matrix (3.3), we have $\Delta F = \operatorname{tr}(g_0^{-1}(\partial_{\zeta^i}\partial_{\overline{\zeta}^j} F))$. Thus, by $(\partial_{\zeta^i}\partial_{\overline{\zeta}^j} U_\varepsilon) = g_0 + \varepsilon\,(\partial_{\zeta^i}\partial_{\overline{\zeta}^j} F) + O(\varepsilon^2)$, we

get

$$\begin{aligned}
\det(\partial_{\zeta^i}\partial_{\overline{\zeta}^j}U_\varepsilon) &= \det\Big(g_0\,(I_{n+1}+\varepsilon\, g_0^{-1}\,(\partial_{\zeta^i}\partial_{\overline{\zeta}^j}F))\Big)+O(\varepsilon^2)\\
&= \det g_0\,\det\Big(I_{n+1}+\varepsilon\, g_0^{-1}\,(\partial_{\zeta^i}\partial_{\overline{\zeta}^j}F)\Big)+O(\varepsilon^2)\\
&= (-1)^n\Big(1+\varepsilon\,\mathrm{tr}(g_0^{-1}\,(\partial_{\zeta^i}\partial_{\overline{\zeta}^j}F))\Big)+O(\varepsilon^2)\\
&= (-1)^n(1+\varepsilon\,\Delta F)+O(\varepsilon^2).
\end{aligned}$$

Noting that $\det(\partial_{\zeta^i}\partial_{\overline{\zeta}^j}U_\varepsilon)$ is independent of ε, we obtain $\Delta F=0$.

(iv) By induction on m, we construct F_m of the form (4.6) satisfying $\Delta F_m = O(\mu^m)$, where $O(\mu^m)$ stands for an expression of the form $\mu^m(\varphi+\psi\log\mu)$ with $\varphi,\psi\in\mathcal{E}(1-m)$. For $m=0$, we may take F_0 to be an arbitrary extension of f to $\mathcal{E}(1)$. Assume we have constructed F_{m-1} with some $m>0$. When $m<n+1$, we set $F_m=F_{m-1}+\varphi_m\,\mu^m$ with $\varphi_m\in\mathcal{E}(1-m)$, and get

$$\begin{aligned}
\Delta F_m &= \Delta F_{m-1}+[\Delta,\mu^m]\varphi_m+\mu^m\Delta\varphi_m\\
&= \Delta F_{m-1}+m\,\mu^{m-1}(Z+\overline{Z}+n+m)\varphi_m+O(\mu^m)\\
&= \Delta F_{m-1}+m(n+2-m)\mu^{m-1}\varphi_m+O(\mu^m).
\end{aligned}$$

Thus $\Delta F_m=0$ holds with $\varphi_m=\mu^{1-m}\Delta F_{m-1}/m(n+2-m)$. When $m\ge n+2$, we set $F_m=F_{m-1}+\mu^m(\varphi_m+\eta_m\log\mu)$ with $\varphi_m,\eta_m\in\mathcal{E}(1-m)$. Using

$$\begin{aligned}
[\Delta,\mu^m\log\mu] = \;& m\mu^{m-1}\log\mu\,(Z+\overline{Z}+n+m)\\
&+\mu^{m-1}(Z+\overline{Z}+2m+n),
\end{aligned}\tag{4.12}$$

we then obtain

$$\begin{aligned}
\Delta F_m &= \Delta F_{m-1}+m(n+2-m)\mu^{m-1}\,(\varphi_m+\eta_m\log\mu)\\
&\quad+(n+2)\mu^{m-1}\eta_m+O(\mu^m).
\end{aligned}$$

If $m=n+2$, then $\Delta F_{n+2}=\Delta F_{n+1}+(n+2)\mu^{n+1}\eta_{n+2}+O(\mu^{n+2})$, so that φ_{n+2} and η_{n+2} are determined by (4.9) and $\Delta F_{n+2}=O(\mu^{n+2})$, respectively. If $m>n+2$, then $n+2-m\ne 0$ and thus η_m,φ_m are uniquely determined by $\Delta F_m=O(\mu^m)$ as in the case $m\le n+1$. This completes the inductive step. The solution to $\Delta F=0$ is then given by the limit of F_m as $m\to\infty$. The uniqueness assertion is clear by the construction. □

Using (iv), we define a linear map

$$\mathcal{L}\colon \mathcal{J}(1)\oplus\mathcal{J}(-n-1)\to\mathcal{E}(1),\tag{4.13}$$

where $\mathcal{L}(f,h)=\varphi$ is the smooth part of the solution to $\Delta(\varphi+\eta\mu^{n+2}\log\mu)=0$ satisfying (4.9). Setting $\varphi=\mathcal{L}(f_A,h_C)$ for $(A,C)\in\mathcal{N}\oplus\mathcal{C}$, we get, by (i) and (ii),

$$r_{\varepsilon\#}=\mu+\varepsilon\,\varphi+O(\varepsilon^2).\tag{4.14}$$

Applying $\partial\overline{\partial}$, we obtain

$$g_\varepsilon = g_0 + \varepsilon\ (\varphi_{i\overline{j}}) + O(\varepsilon^2).$$

Since g_0 is a flat metric, the curvature $R^\varepsilon_{\alpha\overline{\beta}}$ of g_ε satisfies

$$R^\varepsilon_{\alpha\overline{\beta}} = \varepsilon\, \partial_\zeta^\alpha \partial_{\overline{\zeta}}^\beta\, \varphi + O(\varepsilon^2) \quad \text{for} \quad |\alpha|, |\beta| \geq 2, \tag{4.15}$$

where $\partial_\zeta^{i\cdots j} = \partial_{\zeta^i} \cdots \partial_{\zeta^j}$. Hence $d\iota|_0(A, C) = (S_{\alpha\overline{\beta}})$ is given by

$$S_{\alpha\overline{\beta}} = \begin{cases} \partial_\zeta^\alpha \partial_{\overline{\zeta}}^\beta\, \varphi(e_0) & \text{if } |\alpha| \geq 2 \text{ and } |\beta| \geq 2; \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, if we identify $(S_{\alpha\overline{\beta}})$ with a series in $\mathcal{E}_1$, then

$$d\iota|_0(A, C) = \pi \circ \mathcal{L}(f_A, h_C), \tag{4.16}$$

where $\pi\colon \mathcal{E}(1) \to \mathcal{E}_1$ is the projection with respect to the decomposition (4.1). Using this expression, we now prove:

PROPOSITION 4.5. *The map $d\iota|_0$ is injective.*

Proof. Assuming $d\iota|_0(A, C) = 0$ for $(A, C) \in \mathcal{N} \oplus \mathcal{C}$, we shall prove $(A, C) = (0, 0)$. By (4.16), this assumption is equivalent to $\mathcal{L}(f_A, h_C) \in \mathcal{E}^1$. We first write down the set $(\operatorname{Range} \mathcal{L}) \cap \mathcal{E}^1$ explicitly.

LEMMA 4.6. (i) $\operatorname{Range} \mathcal{L} = \widetilde{\mathcal{H}}(1)$, *where*

$$\widetilde{\mathcal{H}}(1) = \Big\{\varphi \in \mathcal{E}(1) : \Delta\varphi = c_n\, \mu^{n+1} \Delta^{n+2}\varphi,\ \ \Delta^{n+3}\varphi = 0\Big\}$$

with $c_n = (-1)^{n+1}\, ((n+1)!)^{-2}$.

(ii) *Let* $\mathcal{H}^1 = \{\varphi \in \mathcal{E}^1 : \Delta\varphi = 0\}$. *Then*

$$\widetilde{\mathcal{H}}(1) \cap \mathcal{E}^1 = \mathcal{L}(\mathcal{J}^1 \oplus \{0\}) = \mathcal{H}^1. \tag{4.17}$$

Proof of Lemma 4.6. (i) For each $\varphi \in \operatorname{Range} \mathcal{L}$, there exists $\eta \in \mathcal{E}(-n-1)$ such that $F = \varphi + \eta\, \mu^{n+2} \log \mu$ satisfies $\Delta F = 0$. Using (4.12), we then get

$$\begin{aligned} \Delta F &= \Delta\varphi + \mu^{n+2} \log \mu \cdot \Delta\eta + [\Delta, \mu^{n+2} \log \mu]\, \eta \\ &= \Delta\varphi + (n+2)\mu^{n+1}\eta + \mu^{n+2} \log \mu \cdot \Delta\eta \end{aligned}$$

so that $\Delta F = 0$ is reduced to a system

$$\begin{cases} \Delta\varphi + (n+2)\mu^{n+1}\eta = 0, \\ \Delta\eta = 0. \end{cases} \tag{4.18}$$

Noting that this system implies $\Delta^{n+2}\varphi = (-1)^n\, (n+1)!\, (n+2)!\eta$, we replace η in (4.18) by $(-1)^n \Delta^{n+2}\varphi / [(n+1)!\, (n+2)!]$. Then

$$\begin{cases} \Delta\varphi = c_n\, \mu^{n+1} \Delta^{n+2}\varphi, \\ \Delta^{n+3}\varphi = 0. \end{cases} \tag{4.19}$$

Thus we have $\operatorname{Range} \mathcal{L} = \widetilde{\mathcal{H}}(1)$.

(ii) We first show that $\widetilde{\mathcal{H}}(1)\cap\mathcal{E}^1\supset\mathcal{L}(\mathcal{J}^1\oplus\{0\})$. Since $\widetilde{\mathcal{H}}(1)=\operatorname{Range}\mathcal{L}$, it suffices to prove $\mathcal{E}^1\supset\mathcal{L}(\mathcal{J}^1\oplus\{0\})$. For $f\in\mathcal{J}^1$, take its extension $\widetilde{f}\in\mathcal{E}^1$ such that $\partial_\mu\widetilde{f}=0$, and set

$$\varphi=\widetilde{f}-\mu\Delta\widetilde{f}/(n+1)\in\mathcal{E}^1.$$

Then, $\varphi|_{\mathcal{Q}}=\widetilde{f}|_{\mathcal{Q}}=f$, $\partial_\mu^{n+2}\varphi|_{\mathcal{Q}}=0$ and

$$(n+1)\Delta\varphi=(n+1)\Delta\widetilde{f}-[\Delta,\mu]\Delta\widetilde{f}-\mu\Delta^2\widetilde{f}=-\mu\Delta^2\widetilde{f}=0,$$

so that $\mathcal{L}(f,0)=\varphi\in\mathcal{E}^1$. We next show $\widetilde{\mathcal{H}}(1)\cap\mathcal{E}^1\subset\mathcal{H}^1$. For $\varphi\in\widetilde{\mathcal{H}}(1)$, we have $\Delta\varphi=c_n\mu^{n+2}\Delta^{n+2}\varphi$, while $\varphi\in\mathcal{E}^1$ implies $\Delta^2\varphi=0$. Therefore, if $\varphi\in\widetilde{\mathcal{H}}(1)\cap\mathcal{E}^1$, then $\Delta\varphi=0$ and thus $\varphi\in\mathcal{H}^1$. It remains to prove $\mathcal{H}^1\subset\mathcal{L}(\mathcal{J}^1\oplus\{0\})$. But this is clear since each $\varphi\in\mathcal{H}^1$ satisfies $\varphi=\mathcal{L}(\varphi|_{\mathcal{Q}},0)$ $\in\mathcal{L}(\mathcal{J}^1\oplus\{0\})$. □

We now return to the proof of Proposition 4.5. By (4.17), there exists an $f_1\in\mathcal{J}^1$ such that $\mathcal{L}(f_A-f_1,h_C)=0$. The injectivity of $\mathcal{L}$ then implies $f_A-f_1=0$ and $h_C=0$. Noting that (4.4) forces $f_A=f_1=0$, we get $(A,C)=(0,0)$ as desired. □

4.3. *Proof of Proposition* 4.1. By virtue of Proposition 4.5, we can apply the inverse function theorem to ι_m, and obtain a neighborhood V of $(0,0)\in[\mathcal{N}]_m\times[\mathcal{C}]_m$ such that $\iota_m|_V$ is an embedding. Noting that ι_m is compatible with the action of δ_t, we see that $\iota_m|_{V_t}$ is also an embedding, where $V_t=\{\delta_t.(A,C):(A,C)\in V\}$. Since $t>0$ is arbitrary, it follows that $\iota_m\colon[\mathcal{N}]_m\times[\mathcal{C}]_m\to[\mathbf{T}_1]_m$ itself is an embedding. Thus $[\mathcal{R}]_m=\iota_m([\mathcal{N}]_m\times[\mathcal{C}]_m)$ is a real-analytic submanifold of $[\mathbf{T}_1]_m$, and there exists a real-analytic inverse map $\iota_m^{-1}\colon[\mathcal{R}]_m\to[\mathcal{N}]_m\times[\mathcal{C}]_m$.

We now construct τ_m inductively. The case $m=0$ is trivial because $[\mathcal{N}]_0\times[\mathcal{C}]_0=\{(0,0)\}$. Assume we have gotten $\tau_{m-1}(T)$. We construct $\tau_m(T)$ as follows. Denote the components of $\iota_m^{-1}(R)$, $R\in[\mathcal{R}]_m$, by

$$(P^l_{\alpha\overline{\beta}}(R),Q^l_{\alpha\overline{\beta}}(R))\in[\mathcal{N}]_m\times[\mathcal{C}]_m.$$

For $(P^l_{\alpha\overline{\beta}}(R),Q^l_{\alpha\overline{\beta}}(R))\in[\mathcal{N}]_{m-1}\times[\mathcal{C}]_{m-1}$, define their polynomial extensions to $[\mathbf{T}_1]_m$ by the components of $\tau_{m-1}(T)=(P^l_{\alpha\overline{\beta}}(T),Q^l_{\alpha\overline{\beta}}(T))$. For the components of weight $>m-1$, we construct their polynomial extensions in two steps. First, extend $P^l_{\alpha\overline{\beta}}(R)$, $Q^l_{\alpha\overline{\beta}}(R)$ to real-analytic functions $\widetilde{P}^l_{\alpha\overline{\beta}}(T)$, $\widetilde{Q}^l_{\alpha\overline{\beta}}(T)$ on $[\mathbf{T}_1]_m$ in such a way that they have homogeneous weight. Next, neglect the monomials of degrees $>m$ in $\widetilde{P}^l_{\alpha\overline{\beta}}(T)$, $\widetilde{Q}^l_{\alpha\overline{\beta}}(T)$ and define polynomials $P^l_{\alpha\overline{\beta}}(T)$, $Q^l_{\alpha\overline{\beta}}(T)$. These polynomials are extensions of $P^l_{\alpha\overline{\beta}}(R)$, $Q^l_{\alpha\overline{\beta}}(R)$ because of the following lemma.

LEMMA 4.7. *A monomial $P(T)$ on $\mathbf{T}_1$ vanishes on $\mathcal{R}$ provided the weight is less than the degree.*

Proof. Let $Q(A,C) = P(\iota(A,C))$ be of weight w. Then, by the assumption on $P(T)$, each monomial constituting $Q(A,C)$ has degree $> w$. But such a monomial must be 0 because all the variables $A^l_{\alpha\overline{\beta}}$ and $C^l_{\alpha\overline{\beta}}$ have weight ≥ 1. Thus we have $Q(A,C) = 0$, which is equivalent to $P(T) = 0$ on $\mathcal{R}$. □

The collection $(P^l_{\alpha\overline{\beta}}(T), Q^l_{\alpha\overline{\beta}}(T))$ gives a polynomial map $\tau_m(T)$ satisfying $\pi_m \circ \tau_m = \tau_{m-1} \circ \pi'_m$ and $\tau_m \circ \iota_m = \mathrm{id}$. This completes the inductive step.

Remark 4.8. Using the method of linearization in this section, we can now prove the statement in Remark 1.1. Given $(A,C) \in \mathcal{N} \oplus \mathcal{C}$, let $u^{\mathrm{G}}_\varepsilon$ be the asymptotic solution to (1.1) of the form (1.3) (with $\eta^{\mathrm{G}}_1 = 1$) for the surface $N(\varepsilon A)$ satisfying (1.12) with εC in place of C. Then $F = |z^0|^2 (du^{\mathrm{G}}_\varepsilon/d\varepsilon)|_{\varepsilon=0}$ can be written in the form

$$F = \varphi + \mu^{n+2}\eta \log(\mu|\zeta|^{-2}), \quad \text{where } \varphi \in \mathcal{E}(1),\ \eta \in \mathcal{E}(-n-1), \tag{4.20}$$

which satisfies (4.7), (4.8) and $\Delta F = 0$. We denote φ in (4.20) by $\varphi[A,C]$, and set $\mathcal{H}^{\mathrm{G}} = \{\varphi[A,C] : (A,C) \in \mathcal{N} \oplus \mathcal{C}\}$. Then, for $\varphi, \widetilde{\varphi} \in \mathcal{H}^{\mathrm{G}}$, $\varphi - \widetilde{\varphi} = O(\mu)$ if and only if $\varphi - \widetilde{\varphi} = \mu^{n+2}\psi$ with $\psi \in \mathcal{E}(-n-1)$ satisfying $\Delta\psi = 0$. Let $\mathcal{F}^{\mathrm{G}}_{\partial\Omega}$ be the space of defining functions in Remark 1.1, and assume that $\mathcal{F}^{\mathrm{G}}_{\partial\Omega}$ satisfies the transformation law (1.6). Linearizing (1.6), we then see that, for each $h \in \mathrm{SU}(g_0)$,

$$\varphi \in \mathcal{H}^{\mathrm{G}} \quad \text{if and only if} \quad \widetilde{\varphi}(\zeta) := \varphi(h\zeta) \in \mathcal{H}^{\mathrm{G}}. \tag{4.21}$$

We next set $\widetilde{F}(\zeta) = F(h\zeta)$ for F in (4.20). Then $\Delta\widetilde{F} = 0$ and $\widetilde{F} = \widehat{\varphi} + \mu^{n+2}\widetilde{\eta}\log(\mu|\zeta^0|^{-2})$, where

$$\widehat{\varphi}(\zeta) = \widetilde{\varphi}(\zeta) + \mu^{n+2}\widetilde{\eta}\log(|\zeta^0/\widetilde{\zeta}^0|^2), \quad \widetilde{\eta}(\zeta) = \eta(\widetilde{\zeta}) \quad (\widetilde{\zeta} = h\zeta). \tag{4.22}$$

Thus $\widetilde{\varphi}, \widehat{\varphi} \in \mathcal{H}^{\mathrm{G}}$ whenever $\varphi \in \mathcal{H}^{G}$. It then follows from $\widehat{\varphi} - \widetilde{\varphi} = O(\mu)$ and (4.22) that $\Delta\widetilde{\eta}\log(|\zeta^0/\widetilde{\zeta}^0|^2) = 0$. But this equation is not satisfied, e.g., by h and φ such that $\widetilde{\zeta}^0 = \zeta^0 + i\zeta^n, \widetilde{\zeta}' = \zeta', \widetilde{\zeta}^n = \zeta^n$ and φ satisfying $\varphi = |\zeta^0|^2|\zeta^1/\zeta^0|^{2(n+2)} + O(\mu)$, in which case $\eta = (-1)^{n+1}(n+2)|\zeta^0|^{-2(n+1)}$. This is a contradiction, and we have proved the statement in Remark 1.1

5. Proof of Theorem 5

5.1. *Linearization.* We have seen in Theorem 4 that $\mathcal{R}$ is a submanifold of $\mathbf{T}_1$ with a system of polynomial defining equations $\iota \circ \tau(T) - T = 0$, where $T = (T_{\alpha\overline{\beta}}) \in \mathbf{T}_1$. Using this fact, we first reduce the study of H-invariants of $\mathcal{R}$ to that of the invariants of the H-module $T_0\mathcal{R}$. That is, we reduce Theorem 5 to the following:

Theorem 5′. *Every H-invariant of $T_0\mathcal{R}$ is the restriction to $T_0\mathcal{R}$ of a linear combination of elementary invariants of* $\mathbf{T}_1$.

Proof of Theorem 5 *using Theorem* 5′. We follow the argument of [F3]. Taking an H-invariant I of $\mathcal{R}$ of weight w, we shall show that, for any N, there exists a finite list of elementary invariants W_j such that I is written in the form

$$I = \sum c_j W_j + Q_N \quad \text{on } \mathcal{R} \quad \text{with } Q_N(T) = O(T^N), \tag{5.1}$$

where $O(T^m)$ stands for a term (polynomial) which does not contain monomials of degree $< m$. Once this is proved, Theorem 5 follows. In fact, by taking N so that $N > w$, we obtain by Lemma 4.7 that $Q_N = 0$ on $\mathcal{R}$, that is, $I = \sum c_j W_j$ on $\mathcal{R}$.

To prove (5.1), we start by writing $I(T) = O(T^m)$ so that

$$I(T) = S^m(T) + O(T^{m+1}),$$

where S^m is homogeneous of degree m. Then S^m is an H-invariant of $T_0\mathcal{R}$. In fact, if we take a curve γ_ε in $\mathcal{R}$ such that $\gamma_0 = 0$ and $(d\gamma_\varepsilon/d\varepsilon)|_{\varepsilon=0} = T \in T_0\mathcal{R}$, then we have $S^m(T) = \lim_{\varepsilon\to 0} I(\gamma_\varepsilon)/\varepsilon^m$. Since the right-hand side is H-invariant, so is S^m as claimed. Therefore, we can find, by using Theorem 5′, elementary invariants W_j such that

$$S^m = \sum c_j W_j + U, \tag{5.2}$$

where U is homogeneous of degree m and vanishes on $T_0\mathcal{R}$. We now examine the remainder U. Let $\{P_i(T)\}_{i=1}^\infty$ be a system of polynomials in the variables $T_{\alpha\overline{\beta}}$ which defines $\mathcal{R}$, i.e., $\mathcal{R} = \cap_{i=1}^\infty \{P_i = 0\}$, and let p_i be the linear part of P_i so that $T_0\mathcal{R} = \cap_{i=1}^\infty \ker p_i$. We write U as a finite sum $U = \sum U_i\, p_i$, where U_i are homogeneous of degree $m-1$. Then

$$U = \sum U_i\,(p_i - P_i) + \sum U_i P_i = \sum U_i\,(p_i - P_i) \quad \text{on } \mathcal{R}.$$

Noting $\sum U_i\,(p_i - P_i) = O(T^{m+1})$ and using (5.2), we obtain (5.1) for $N = m+1$. Repeating the same procedure for the remainder Q_{m+1}, we obtain the expression (5.1) inductively for arbitrary N. □

5.2. *A short exact sequence characterizing* $T_0\mathcal{R}$. We further reduce Theorem 5′ to an analogous theorem for a simpler H-module of trace-free tensors. This is done by writing down a system of equations of $T_0\mathcal{R}$ explicitly and giving a short exact sequence which characterizes $T_0\mathcal{R}$, where $T_0\mathcal{R} = d\iota|_0(\mathcal{N}\oplus\mathcal{C})$ is regarded as a subspace of $\mathcal{E}_1$ as in subsection 4.2.

Proposition 5.1. (i) *The tangent space $T_0\mathcal{R}$ of $\mathcal{R}$ at* 0 *is given by*

$$\widetilde{\mathcal{H}}_1 := \widetilde{\mathcal{H}}(1) \cap \mathcal{E}_1 = \left\{\varphi \in \mathcal{E}_1 : \Delta\varphi = c_n\,\mu^{n+1}\Delta^{n+2}\varphi,\ \Delta^{n+3}\varphi = 0\right\}.$$

(ii) *The following sequence is exact*:

$$0 \to \mathcal{H}_1 \hookrightarrow \widetilde{\mathcal{H}}_1 \xrightarrow{\Delta^{n+2}} \mathcal{H}(-n-1) \to 0, \tag{5.3}$$

where $\mathcal{H}(k) = \{\varphi \in \mathcal{E}(k) : \Delta\varphi = 0\}$ *and* $\mathcal{H}_k = \mathcal{H}(k) \cap \mathcal{E}_k$.

Proof. (i) Since (4.17) implies $\pi \circ \mathcal{L}(\mathcal{J}^1 \oplus \{0\}) = \{0\}$, it follows from (4.4) that

$$\pi \circ \mathcal{L}\left(\mathcal{N} \oplus \mathcal{J}(-n-1)\right) = \pi \circ \mathcal{L}\left(\mathcal{J}(1) \oplus \mathcal{J}(-n-1)\right).$$

Using Lemma 4.6 (i), we then get

$$\begin{aligned} T_0\mathcal{R} &= \pi \circ \mathcal{L}\left(\mathcal{N} \oplus \mathcal{J}(-n-1)\right) \\ &= \pi(\operatorname{Range}\mathcal{L}) \\ &= \pi(\widetilde{\mathcal{H}}(1)) = \widetilde{\mathcal{H}}_1. \end{aligned}$$

(ii) It is clear from the definition of $\widetilde{\mathcal{H}}_1$ that $0 \to \mathcal{H}_1 \hookrightarrow \widetilde{\mathcal{H}}_1 \xrightarrow{\Delta^{n+2}} \mathcal{H}(-n-1)$ is exact. It then remains only to prove the surjectivity of Δ^{n+2}. To show this, we solve the equation

$$\Delta^{n+2}\varphi = \eta \quad \text{for } \eta \in \mathcal{H}(-n-1) \text{ given.} \tag{5.4}$$

We need to find a solution $\varphi \in \widetilde{\mathcal{H}}_1$. But it suffices to find φ in $\widetilde{\mathcal{H}}(1) = \widetilde{\mathcal{H}}_1 \oplus \mathcal{H}^1$, because all $\varphi \in \mathcal{H}^1$ satisfy $\Delta^{n+2}\varphi = 0$. Next, we follow the argument of [EG2, Prop. 4.5]. We first recall a lemma in [EG1].

LEMMA 5.2. *For $k \le 0$, the restriction $\mathcal{H}(k) \ni \eta \mapsto \eta|_{\mathcal{Q}} \in \mathcal{J}(k)$ is an isomorphism.*

Proof. This amounts to proving the unique existence of a solution $\eta \in \mathcal{H}(k)$ to the equation $\Delta\eta = 0$ under the condition $\eta|_{\mathcal{Q}} = f \in \mathcal{J}(k)$. The proof is a straightforward modification of that of our Proposition 4.3, (iv). □

By this lemma, we can reduce (5.4) to an equation for $f \in \mathcal{J}(1)$:

$$\Delta^{n+2}\mathcal{L}(f, 0)|_{\mathcal{Q}} = g, \quad \text{where } g = \eta|_{\mathcal{Q}} \in \mathcal{J}(-n-1). \tag{5.5}$$

We write down the left-hand side with the real coordinates $(t, x) = (t, x^1, \dots, x^{2n})$ of $\mathcal{Q}$, where $2\zeta^0 = t + i\,x^1$, $2\zeta^j = x^{2j} + i\,x^{2j+1}$, $j = 1, \dots, n-1$, and $x^{2n} = 2\,\mathrm{Im}\zeta^n$.

LEMMA 5.3. *Let φ be a formal power series about $e_0 \in \mathbb{C}^{n+1}$ of homogeneous degree 2 in the sense that $(Z + \overline{Z})\varphi = 2\varphi$. Then*

$$(\Delta^{n+2}\varphi)|_{\mathcal{Q}} = \Delta_x^{n+2}(\varphi|_{\mathcal{Q}}), \quad \textit{where } \Delta_x = 2\partial_{x^1}\partial_{x^{2n}} - \sum_{j=2}^{2n-1} \partial_{x^j}^2. \tag{5.6}$$

Proof. In terms of the coordinates (t, x, μ), the Laplacian Δ is written as

$$\Delta = \Delta_x + (\mu\partial_\mu + E + n + 1)\,\partial_\mu, \quad \text{where } E = t\partial_t + \sum_{j=1}^{2n} x^j \partial_{x^j}.$$

Writing $\varphi(t, x, \mu) = \varphi_0(t, x) + \mu\,\psi(t, x, \mu)$, we have

$$\Delta^{n+2}\varphi = \Delta_x^{n+2}\varphi_0 + \Delta^{n+2}(\mu\,\psi).$$

Noting that ψ is homogeneous of degree 0, we have

$$\begin{aligned}\Delta^{n+2}(\mu\,\psi) &= [\Delta^{n+2}, \mu]\psi + O(\mu) \\ &= (n+2)(Z + \overline{Z} + 2n + 2)\Delta^{n+1}\psi + O(\mu) \\ &= O(\mu).\end{aligned}$$

Therefore, $\Delta^{n+2}\varphi = \Delta_x^{n+2}\varphi_0 + O(\mu)$, which is equivalent to (5.6). $\square$

Since (5.6) implies $\Delta^{n+2}\mathcal{L}(f, 0)|_{\mathcal{Q}} = \Delta_x^{n+2} f$, we can reduce (5.5) to

$$\Delta_x^{n+2} f = g. \tag{5.7}$$

It is a standard fact of harmonic polynomials that, for each polynomial $q(x)$ of homogeneous degree k, there exists a polynomial $p(x)$ of homogeneous degree k such that

$$\Delta_x^{n+2}\mu_x^{n+2} p = q, \quad \text{where } \mu_x = 2x^1 x^{2n} - \sum_{j=2}^{2n-1}(x^j)^2.$$

We apply this fact to solving (5.7). Writing $g(t, x) = \sum_{j=2n+2}^{\infty} q_j(x)\,t^{-j}$ with polynomials $q_j(x)$ of homogeneous degree $j - 2n - 2$, we take, for each j, a polynomial $p_j(x)$ such that $\Delta_x^{n+2}\mu_x^{n+2} p_j = q_j$. Then

$$\Delta_x^{n+2}\widetilde{f} = g, \quad \text{where } \widetilde{f} = \mu_x^{n+2}\sum_{j=2n+2}^{\infty} p_j(x)\,t^{-j}.$$

It is clear that $\widetilde{f}$ is homogeneous of degree 2, though $\widetilde{f}$ may not be contained in $\mathcal{J}(1)$. Let us write $\widetilde{f} = \sum_{p+q=2} f^{(p,q)}$, where $f^{(p,q)}$ is homogeneous of degree (p, q). We then see by (5.6) that $\Delta_x^{n+2} f^{(p,q)}$ is homogeneous of degree $(p - n - 2, q - n - 2)$. Setting $f = f^{(1,1)} \in \mathcal{J}(1)$, we thus obtain (5.7). The proof of Proposition 5.1 is complete. $\square$

Now we use the exact sequence (5.3) and reduce Theorem 5′ to:

Theorem $5''$. *Every H-invariant of $\mathcal{H}_1 \oplus \mathcal{H}(-n-1)$ is realized by the restriction to $\mathcal{H}_1 \oplus \mathcal{H}(-n-1)$ of a linear combination of elementary invariants of $\mathcal{E}_1 \oplus \mathcal{E}(-n-1)$. Here, an elementary invariant of $\mathcal{E}_1 \oplus \mathcal{E}(-n-1)$ is defined to be a complete contraction of the form*

$$\operatorname{contr}\left(R^{(p_1,q_1)} \otimes \cdots \otimes R^{(p_d,q_d)} \otimes E^{(p'_1,q'_1)} \otimes \cdots \otimes E^{(p'_{d'},q'_{d'})}\right),$$

where $R^{(p,q)} = (R_{\alpha\overline{\beta}})_{|\alpha|=p,|\beta|=q}$ and $E^{(p,q)} = (E_{\alpha\overline{\beta}})_{|\alpha|=p,|\beta|=q}$ with $(R_{\alpha\overline{\beta}}, E_{\alpha\overline{\beta}}) \in \mathcal{E}_1 \oplus \mathcal{E}(-n-1) \subset \mathbf{T}_1 \oplus \mathbf{T}_{-n-1}$.

Proof of Theorem $5'$ using Theorem $5''$. We embed $T_0\mathcal{R}$ into $\mathcal{E}_1 \oplus \mathcal{E}(-n-1)$ as a subspace $\mathcal{H} = \{(R, \Delta^{n+2}R) : R \in T_0\mathcal{R}\}$ by identifying $R = (R_{\alpha\overline{\beta}})$ with a formal power series in $\mathcal{E}_1$. We wish to find, for any N, a list of elementary invariants $\{W_j\}$ of $\mathbf{T}_1$ such that I is written in the form

$$I = \sum c_j W_j + O(E^N) \quad \text{on } \mathcal{H}, \tag{5.8}$$

where $O(E^N)$ is a polynomial in $(R_{\alpha\overline{\beta}}, E_{\alpha\overline{\beta}})$ consisting of monomials of degree $\geq N$ in E. If $(n+1)N$ is greater than the weight of I, then the error term $O(E^N)$ vanishes, because each component $E_{\alpha\overline{\beta}}$ has weight $\geq n+1$, and the reduction to showing (5.8) is done.

The proof of (5.8) goes as an analogy of the procedure of linearization. Writing

$$I(R,E) = S^m(R,E) + O(E^{m+1}),$$

where S^m is homogeneous of degree m in E, we show that S^m is an H-invariant of $\mathcal{H}_1 \oplus \mathcal{H}(-n-1)$. For any $(R,E) \in \mathcal{H}_1 \oplus \mathcal{H}(-n-1)$, we use (5.3) and choose $\widetilde{R}$ such that $(\widetilde{R}, E) \in \mathcal{H}$. Then we have $S^m(R,E) = \lim_{\varepsilon \to 0} I(R + \varepsilon\widetilde{R}, \varepsilon E)/\varepsilon^m$. Since the right-hand side is H-invariant, so is S^m as claimed. Therefore we can find, by Theorem $5''$, a list of elementary invariants $W_j(R,E)$ of $\mathcal{E}_1 \oplus \mathcal{E}(-n-1)$ and write S^m as

$$S^m = \sum c_j W_j + U,$$

where U vanishes on $\mathcal{H}_1 \oplus \mathcal{H}(-n-1)$ and is homogeneous of degree m in E. Note that each elementary invariant $W_j(R,E)$ coincides on $\mathcal{H}$ with the elementary invariant $W_j(R, \Delta^{n+2}R)$ of $\mathbf{T}_1$. We next study the remainder U. Recall by Proposition 5.1 that $\mathcal{H}$ is written as

$$\mathcal{H} = \{(R,E) : P_i(R) = \widetilde{Q}_i(E), Q_i(E) = 0 \text{ and } E = \Delta^{n+2}R\},$$

where $\{P_i(R)\}_{i=1}^\infty$ and $\{Q_i(E), \widetilde{Q}_i(E)\}_{i=1}^\infty$ are systems of linear functions on $\mathcal{E}_1$ and $\mathcal{E}(-n-1)$, respectively, such that

$$\mathcal{H}_1 = \cap_i \ker P_i \quad \text{and} \quad \mathcal{H}(-n-1) = \cap_i \ker Q_i.$$

Using the defining functions $\{P_i, Q_i\}$ of $\mathcal{H}_1 \oplus \mathcal{H}(-n-1)$, we can express U as a finite sum $U = \sum U_i P_i + \sum V_i Q_i$, where U_i (resp. V_i) are homogeneous of

degree m (resp. $m-1$) in E. We thus write U in the form

$$U = \sum U_i \widetilde{Q}_i + \sum U_i(P_i - \widetilde{Q}_i) + \sum V_i Q_i \tag{5.9}$$

and find that $U = \sum U_i \widetilde{Q}_i = O(E^{m+1})$ on $\mathcal{H}$, because the last two sums in (5.9) vanish on $\mathcal{H}$. We have shown (5.8) for $N = m+1$. Repeating the same procedure for the remainder, we obtain (5.8) for arbitrary N. □

5.3. *Proof of Theorem* 5″. Since H acts on $\mathcal{H}_1 \oplus \mathcal{H}(-n-1)$ by linear transformations, we may restrict our attention to the H-invariants $I(R,E)$ which are homogeneous of degrees d_R and d_E in R and E, respectively. If $d_E = 0$, we may regard $I(R,E)$ as an invariant $I(R)$ of $\mathcal{H}_1$. For $I(R)$, we can apply Theorem C of [BEG] and express it as a linear combination of elementary invariants of $\mathcal{H}_1$. We thus assume $d_E \geq 1$, and again follow the arguments of [BEG].

The first step of the proof is to express $I(R,E)$ as a component of a linear combination of partial contractions. We denote by $\odot^{p,q}W^*$ the space of bisymmetric tensors of type (p,q) on W^* and by $\odot_0^{p,q}W^*$ the subspace of $\odot^{p,q}W^*$ consisting of trace-free tensors. Let e^* be the row vector $(0,\dots,0,1)$ $\in W^* \otimes \sigma_{1,0}$. Then we have:

Lemma 5.4. *For some integer $m \leq w - d_R - (n+1)d_E$, a map*

$$C\colon \mathcal{H}_1 \oplus \mathcal{H}(-n-1) \to \odot_0^{m,m}W^* \otimes \sigma_{m-w,m-w}$$

is defined by making a linear combination of partial contractions of the tensors $R^{(p,q)}, E^{(p,q)}$ and e^*, $\overline{e}^*$ such that

$$C_{n\cdots n\,\overline{n}\cdots\overline{n}} = I. \tag{5.10}$$

Proof. Since $R_{\alpha\overline{\beta}}$ and $E_{\alpha\overline{\beta}}$ satisfy

$$\begin{aligned} R_{\alpha 0\overline{\beta}} &= (1-|\alpha|)R_{\alpha\overline{\beta}}, & R_{\alpha\,\overline{\beta}\,\overline{0}} &= (1-|\beta|)R_{\alpha\,\overline{\beta}}, \\ E_{\alpha 0\overline{\beta}} &= (-n-1-|\alpha|)E_{\alpha\overline{\beta}}, & E_{\alpha\,\overline{\beta}\,\overline{0}} &= (-n-1-|\beta|)E_{\alpha\,\overline{\beta}}, \end{aligned} \tag{5.11}$$

we can write $I(R,E)$ as a polynomial in the components of the form

$$\widehat{R}^{k\overline{l}}_{\alpha\overline{\beta}} = R_{\alpha\underbrace{n\cdots n}_{k}\overline{\beta}\,\underbrace{\overline{n}\cdots\overline{n}}_{l}}, \quad \widehat{E}^{k\overline{l}}_{\alpha\overline{\beta}} = E_{\alpha\underbrace{n\cdots n}_{k}\overline{\beta}\,\underbrace{\overline{n}\cdots\overline{n}}_{l}},$$

where α,β are lists of indices between 1 and $n-1$. We now regard $\widehat{R}^{k\overline{l}}_{\alpha\overline{\beta}}, \widehat{E}^{k\overline{l}}_{\alpha\overline{\beta}}$ as tensors on $\mathbb{C}^{n-1}$ by fixing $k,\overline{l}$. For these tensors, the Levi factor

$$L = \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & 1/\overline{\lambda} \end{pmatrix} : u \in \mathrm{U}(n-1),\ \lambda\overline{\lambda}^{-1}\det u = 1 \right\}$$

of H acts as the usual tensorial action of u up to a scale factor depending on λ. Thus we may regard $I(R,E)$ as a $\mathrm{U}(n-1)$-invariant polynomial. By Weyl's classical invariant theory for $\mathrm{U}(n-1)$, we then see that I is expressed as a linear combination of complete contractions of the tensors $\widehat{R}^{k\bar{l}}_{\alpha\bar{\beta}}, \widehat{E}^{k\bar{l}}_{\alpha\bar{\beta}}$ for the standard metric $\delta^{i\bar{\jmath}}$ on $\mathbb{C}^{n-1}$. (See Lemma 7.4 of [BEG] for details of this discussion.)

We next replace the contractions with the metric $\delta^{i\bar{\jmath}}$ by those with the metric g_0. This is done by using the relation

$$\sum_{j=1}^{n-1} T_{j\bar{\jmath}} = -\sum_{i,j=0}^{n} g_0^{i\bar{\jmath}} T_{i\bar{\jmath}} + T_{0\bar{n}} + T_{n\bar{0}} \quad \text{for} \quad (T_{i\bar{\jmath}}) \in W^* \otimes \overline{W}^*.$$

Then several 0 and $\overline{0}$ come out as indices. These can be eliminated by using (5.11), and there remain only n and $\overline{n}$ as indices. We then get an expression of I as a linear combination

$$I = \sum_{j=1}^{k} c_j C^{(j)}_{\underbrace{n\cdots n}_{p_j}\underbrace{\overline{n}\cdots\overline{n}}_{q_j}},$$

where each $C^{(j)} \in \mathbf{T}^{p_j,q_j}_w$ is given by partial contraction of the tensor products of several $R^{(p,q)}$ and $E^{(p,q)}$. In general, p_j, q_j $(1 \le j \le k)$ are different. Denoting by m the maximum of p_j, q_j $(1 \le j \le k)$, we define a tensor

$$C' = \sum_{j=1}^{k} c_j (\otimes^{m-p_j} e^*) \otimes C^{(j)} \otimes (\otimes^{m-q_j} \overline{e}^*) \in \mathbf{T}^{m,m}_w.$$

Then we have $I = C'_{n\cdots n\overline{n}\cdots\overline{n}}$ because $e^*_n = \overline{e}^*_{\overline{n}} = 1$. The map C is now given by taking the trace-free bisymmetric part of C'.

To obtain the estimate $m \le w - d_R - (n+1)d_E$, we note that C contains at least one partial contraction which has no e^* or no $\overline{e}^*$. If such a term consists of $R^{(p_j,q_j)}$, $E^{(p'_j,q'_j)}$ and several e^* (resp. $\overline{e}^*$), then C takes values in $\odot_0^{m,m} W^* \otimes \sigma_{\kappa,\kappa}$ with $\kappa = \sum_{j=1}^{d_R}(1-q_j) + \sum_{j=1}^{d_E}(-n-1-q'_j)$ (resp. the same relation with p in place of q). Hence noting $\kappa = m - w$ and $p_j, q_j \ge 2$, we obtain $m - w \le \sum_{j=1}^{d_R}(-1) + \sum_{j=1}^{d_E}(-n-1)$, which is equivalent to the desired estimate for m. $\square$

Now we regard $\mathcal{H}_1 \oplus \mathcal{H}(-n-1)$ as the space of pairs of formal power series (φ, η) about $e_0 \in W$ and write $I(\varphi,\eta)$ for $I(R,E)$ and $C(\varphi,\eta)$ for $C(R,E)$. If we replace the tensors $R_{\alpha\overline{\beta}}$ (resp. $E_{\alpha\overline{\beta}}$) in the partial contractions in C by the formal power series $\partial^\alpha_\zeta \partial^\beta_{\overline{\zeta}} \varphi$ (resp. $\partial^\alpha_\zeta \partial^\beta_{\overline{\zeta}} \eta$), we obtain a formal power series about $e_0 \in W$ which takes values in $\odot_0^{m,m} W^*$. Restricting this power series to $\mathcal{Q}$ and raising all indices by using g_0, we obtain a map

$$\widetilde{C}\colon \mathcal{H}_1 \oplus \mathcal{H}(-n-1) \to \odot_0^{m,m} W \otimes \mathcal{J}(m-w)$$

which satisfies $\widetilde{C}(\varphi,\eta)|_{e_0} = C(\varphi,\eta)$ when all indices are raised.

Note that $\mathcal{H}(k)$, $\mathcal{H}^k$, $\mathcal{H}_k$ and $\mathcal{J}(k)$ admit a natural structure of $(\mathfrak{su}(g_0), H)$-modules, where $\mathfrak{su}(g_0)$ is the Lie algebra of $\mathrm{SU}(g_0)$. For $\mathcal{H}(k)$, $\mathcal{H}^k$ and $\mathcal{J}(k)$ there are natural $(\mathfrak{su}(g_0), H)$-actions induced from the action of $\mathrm{SU}(g_0)$ on W and $\mathcal{Q}$. For $\mathcal{H}_k$, a $(\mathfrak{su}(g_0), H)$-action is induced via the H-isomorphism $\mathcal{H}_k \cong \mathcal{H}(k)/\mathcal{H}^k$. We also consider the complexification of these spaces and denote them, e.g., by $\mathcal{H}^{\mathbb{C}}(k)$, $\mathcal{J}^{\mathbb{C}}(k)$. Now we have:

Lemma 5.5. (i) *There exists a unique* $(\mathfrak{su}(g_0), H)$*-equivariant map*

$$\widetilde{I}\colon \mathcal{H}_1 \oplus \mathcal{H}(-n-1) \to \mathcal{J}^{\mathbb{C}}(-w)$$

such that $\widetilde{I}(\varphi,\eta)|_{e_0} = I(\varphi,\eta)$ *for any* $(\varphi,\eta) \in \mathcal{H}_1 \oplus \mathcal{H}(-n-1)$.

(ii) *For any* $(\varphi,\eta) \in \mathcal{H}_1 \oplus \mathcal{H}(-n-1)$,

$$\widetilde{C}(\varphi,\eta)^{\alpha\overline{\beta}} = \zeta^{\alpha_1}\cdots\zeta^{\alpha_m}\overline{\zeta}^{\beta_1}\cdots\overline{\zeta}^{\beta_m}\widetilde{I}(\varphi,\eta). \tag{5.12}$$

The proof of this lemma goes exactly the same way as those of Propositions 8.1 and 8.5 of [BEG], where the $(\mathfrak{su}(g_0), H)$-equivariance of $\widetilde{C}$ is used essentially.

The final step in the proof of Theorem $5''$ is to obtain an explicit expression of $\widetilde{I}$ in terms of $\widetilde{C}$ by differentiating both sides of the equation (5.12). We first introduce differential operators $D_{i\overline{j}}\colon \mathcal{E}^{\mathbb{C}}(s+1) \to \mathcal{E}^{\mathbb{C}}(s)$ for $(n+2s)(n+2s+1) \neq 0$ by

$$D_{i\overline{j}} f = \left(\partial_{\zeta^i} - \frac{\zeta_i \Delta}{n+2s}\right)\left(\partial_{\overline{\zeta}^j} - \frac{\overline{\zeta}_j \Delta}{n+2s+1}\right) f,$$

where the index for ζ is lowered with g_0. Then one can easily check the following facts: (i) $D_{i\overline{j}}(\mu f) = O(\mu)$, so that $(D_{i\overline{j}} f)|_{\mathcal{Q}}$ depends only on $f|_{\mathcal{Q}}$; (ii) For any $f \in \mathcal{E}^{\mathbb{C}}(s)$,

$$D_{i\overline{j}}(\zeta^i\overline{\zeta}^j f) = c_s\, f, \quad \text{where} \quad c_s = \frac{(n+s)^2(n+2s+2)}{n+2s}$$

and the repeated indices are summed over $0, 1, \ldots, n$; see Lemma 8.7 of [BEG]. In view of (i), (ii) and taking an arbitrary extension of $\widetilde{I}(\varphi,\eta)$ to $\mathcal{E}(-w)$, we get

$$\begin{aligned} D_{\alpha_1\overline{\beta}_1} D_{\alpha_2\overline{\beta}_2} & \cdots D_{\alpha_m\overline{\beta}_m} \widetilde{C}^{\alpha\overline{\beta}}(\varphi,\eta) \\ &= D_{\alpha_1\overline{\beta}_1} D_{\alpha_2\overline{\beta}_2} \cdots D_{\alpha_m\overline{\beta}_m} (\zeta^\alpha\overline{\zeta}^\beta \widetilde{I}(\varphi,\eta)) \\ &= c_{-w} c_{-w+1} \cdots c_{-w+m-1}\, \widetilde{I}(\varphi,\eta) \quad \text{on } \mathcal{Q}. \end{aligned}$$

Since $m - w \le -d_R - (n+1)d_E$ and $1 \le d_E$, we have $m - w \le -n-1$. Hence, all $D_{i\overline{j}}$ appearing above are well-defined and all $c_s \neq 0$. Therefore,

$$I(\varphi,\eta) = \frac{1}{c_{-w}\cdots c_{-w+m-1}} D_{\alpha_1\overline{\beta}_1}\cdots D_{\alpha_m\overline{\beta}_m} \widetilde{C}^{\alpha\overline{\beta}}(\varphi,\eta)|_{e_0},$$

and I is expressed as a linear combination of complete contractions.

Remark 5.6. The assumption $d_E \geq 1$ was used only in the final step of the proof to ensure $c_s \neq 0$. The argument above is valid even if $d_E = 0$ as long as $d_R \geq n$. This is exactly the proof of Theorem C of [BEG] for invariants of high degrees. To treat the invariants of low degrees on $\mathcal{H}_1$, the authors used an entirely different argument.

Remark 5.7. The tensors $E^{(p,q)}$ used in this section are modeled on the biholomorphically invariant tensors

$$E_k^{(p,q)} = \nabla^p \overline{\nabla}^q (|z^0|^{-2k(n+1)} \eta_k),$$

which were introduced by Graham [G2]. He used these tensors to construct CR invariants from the complete contractions of the form

$$\operatorname{contr}\Big(R^{(p_1,q_1)} \otimes \cdots R^{(p_d,q_d)} \otimes E_{k_1}^{(p'_1,q'_1)} \otimes \cdots E_{k_{d'}}^{(p'_{d'},q'_{d'})}\Big).$$

Such complete contractions give rise to CR invariants if, for example, $p_j, q_j < n+2$ and $p'_j, q'_j < n+1$. This class of CR invariants correspond to $\mathcal{C}$-independent Weyl invariants which contain the covariant derivatives of the Ricci tensor. In fact, $g[r]$ is Ricci flat if and only if $\eta_k = 0$ for all $k \geq 1$, because the Ricci form of $g[r]$ is given by $\partial\overline{\partial} \log J[r]$. Thus, a CR invariant depending on $E_k^{(p,q)}$ must contain the covariant derivatives of the Ricci tensor when it is expressed as a Weyl invariant.

6. Proof of Theorem 3

By virtue of Theorem 2, it suffices to prove:

Proposition 6.1. *Let* $n \geq 3$ (*resp.* $n = 2$). *Then every Weyl invariant of weight* $w \leq n+2$ (*resp.* $w \leq 5$) *is* $\mathcal{C}$-*independent. For* $w = n+3$ (*resp.* $w = 6$), *there exists a* $\mathcal{C}$-*dependent Weyl invariant of weight* w.

Proof of Proposition 6.1. Take a Weyl polynomial $W_\#$ of weight w and set $I(A,C) = I_W(A,C)$. We begin by inspecting the linear part of $I(A,C)$.

Lemma 6.2. *If* $I(A,C)$ *has nonzero linear part, then* $w = n+2$ *and the linear part is a constant multiple of* $\Delta_x^{n+2} f_A(e_0)$ *with* f_A *as in* (4.7).

Proof. If $W_\#(R)$ has no linear terms, neither does $I(A,C)$. Thus it suffices to consider the case where $W_\#(R)$ is a linear complete contraction $\operatorname{contr}(R^{(p,p)})$. In this case, (4.15) implies

$$I(A,C) = \Delta^p \varphi(e_0) + Q(A,C), \tag{6.1}$$

where $Q(A,C)$ is a polynomial in $(A^l_{\alpha\overline{\beta}}, C^l_{\alpha\overline{\beta}})$ without linear terms. By (4.19) and (5.6), $\Delta^p \varphi(e_0) = 0$ if $p \neq n+2$ and $\Delta^{n+2}\varphi(e_0) = -\Delta_x^{n+2} f_A(e_0)$. Thus we obtain the lemma. □

We next consider nonlinear terms in $I(A,C)$. Since $I(A,C)$ is invariant under the action of $\mathrm{U}(n-1)$, it is written as a linear combination of complete contractions of the form

$$\operatorname{contr}'\left(\mathbf{A}^{l_1}_{p_1\overline{q}_1}\otimes\cdots\otimes\mathbf{A}^{l_d}_{p_d\overline{q}_d}\otimes\mathbf{C}^{l'_1}_{p'_1\overline{q}'_1}\otimes\cdots\otimes\mathbf{C}^{l'_{d'}}_{p'_{d'}\overline{q}'_{d'}}\right) \tag{6.2}$$

with $\sum_{j=1}^{d}(p_j+q_j+2l_j-2)+\sum_{j=1}^{d'}(p'_j+q'_j+2l'_j+2n+2)=2w$. Here $\mathbf{C}^l_{p\overline{q}}=(C^l_{\alpha\overline{\beta}})_{|\alpha|=p,|\beta|=q}$ is regarded as a tensor of type (p,q) on $\mathbb{C}^{n-1}$ and the contraction is taken with respect to $\delta^{i\overline{j}}$ for some pairing of lower indices. Suppose (6.2) is nonlinear and contains the variables $C^l_{\alpha\overline{\beta}}$, so that $d+d'\geq 2$ and $d'\geq 1$. Then $p_j+q_j\geq 4$ implies $w\geq n+2$. The equality $w=n+2$ holds only for $\operatorname{contr}'(\mathbf{A}^0_{2\overline{2}}\otimes\mathbf{C}^0_{0\overline{0}})$, while by (N2),

$$\operatorname{contr}'(\mathbf{A}^0_{2\overline{2}}\otimes\mathbf{C}^0_{0\overline{0}})=\operatorname{contr}'(\mathbf{A}^0_{2\overline{2}})C^0=0,$$

where C^0 is the only one component of $\mathbf{C}^0_{0\overline{0}}$. Thus $I(A,C)$ containing $C^l_{\alpha\overline{\beta}}$ has weight $\geq n+3$. In case $w=n+3$, there are only two types of contractions of the form (6.2), namely,

$$\operatorname{contr}'(\mathbf{A}^0_{2\overline{2}}\otimes\mathbf{A}^0_{2\overline{2}}\otimes\mathbf{C}^0_{0\overline{0}})\quad\text{and}\quad\operatorname{contr}'(\mathbf{A}^l_{p\overline{q}}\otimes\mathbf{C}^{l'}_{p'\overline{q}'}), \tag{6.3}$$

where $p+p'=q+q'=3-l-l'$. The contractions of the second type always vanish by (N2) (see §3), and the first ones vanish except for the case $\|\mathbf{A}^0_{2\overline{2}}\|^2\,C^0=\sum_{i,j,k,l=1}^{n-1}|A^0_{ij\,\overline{k}\,\overline{l}}|^2\,C^0$; this also vanishes for $n=2$ because $A^0_{11\,\overline{11}}=\operatorname{tr}\mathbf{A}^0_{2\overline{2}}=0$. This completes the proof of the first statement of Proposition 6.1.

To prove the second statement, we consider, for $n=2$, a complete contraction of weight 6:

$$W_2=\sum_{|\alpha|=6,|\beta|=2}R_{\alpha\overline{\beta}}R^{\overline{\beta}\alpha}$$

and, for $n\geq 3$,

$$W_n=\sum_{|\alpha|=|\beta|=2,|\gamma|=n+2}R_{\alpha\,\overline{\beta}}R^{\overline{\beta}\gamma}R_{\gamma}{}^{\alpha},$$

which has weight $n+3$. Here indices are raised by using g_0. These complete contractions give $\mathcal{C}$-dependent Weyl invariants. In fact:

Lemma 6.3. *Let $I_n(A,C)=I_{W_n}(A,C)$. Then*

$$I_2(A,C)=72\cdot 6!\,(C^0)^2+Q_2(A,C), \tag{6.4}$$

where Q_2 is a polynomial in $(A^l_{\alpha\overline{\beta}},C^l_{\alpha\overline{\beta}})$ such that $Q_2(0,C)=0$. For $n\geq 3$,

$$I_n(A,C)=(-1)^n 64\,(n+2)!\,\|\mathbf{A}^0_{2\overline{2}}\|^2C^0+Q_n(A), \tag{6.5}$$

where $Q_n(A)$ is a polynomial in $A^l_{\alpha\overline{\beta}}$.

Proof. We first prove (6.5). Since $I_n(A,C) = c_n\,\|\mathbf{A}^0_{2\overline{2}}\|^2 C^0 + Q_n(A)$ for some constant c_n, to determine c_n, we consider a family of surfaces with real parameter s

$$2u = |z'|^2 + f_s(z',\overline{z}'), \quad \text{where} \;\; f_s(z',\overline{z}') = 2s\,\mathrm{Re}\,(z^1)^2(\overline{z}^2)^2.$$

Let $A_s \in \mathcal{N}$ denote the list of normal form coefficients of this surface and $C_t \in \mathcal{C}$ the element such that $C^0 = t$ and all the other components vanish. Then

$$I_n(A_s, C_t) = c_n\, s^2 t + Q_n(A_s) = c_n\, s^2 t + O(s^{n+3}). \tag{6.6}$$

On the other hand, we see by (4.15) that the components $R_{\alpha\overline{\beta}}(s,t)$ of $\iota(A_s, C_t)$ satisfy

$$R_{\alpha\overline{\beta}}(s,t) = S_{\alpha\overline{\beta}} + O(s^2+t^2),$$

where $S_{\alpha\overline{\beta}} = \partial_\zeta^\alpha \partial_{\overline{\zeta}}^\beta \varphi(e_0)$ with $\varphi = \mathcal{L}(f_s, t)$. Thus

$$I_n(A_s, C_t) = W'_n + O((s^2+t^2)^2), \tag{6.7}$$

where

$$W'_n = \sum_{|\alpha|=|\beta|=2, |\gamma|=n+2} S_{\alpha\overline{\beta}}\, S^{\overline{\beta}\gamma}\, S_\gamma{}^\alpha. \tag{6.8}$$

Comparing (6.6) with (6.7), we get

$$W'_n = c_n\, s^2 t.$$

Since

$$\varphi = -|\zeta^0|^2 f_s + t\,|\zeta^0|^{-2(n+1)} \mu^{n+2}/(n+2)!,$$

the term $S_{\alpha\overline{\beta}}$ in the sum of (6.8) vanishes except for $S_{11\overline{22}} = S_{22\overline{11}} = -4s$. Using

$$\sum_{|\gamma|=n+2} S^{\overline{22}\,\gamma} S_\gamma{}^{11} = \sum_{|\gamma|=n+2} S_{\overline{11}\,\gamma} S^\gamma{}_{22} = \overline{\sum_{|\gamma|=n+2} S^{\overline{11}\,\gamma} S_\gamma{}^{22}},$$

we then obtain

$$\begin{aligned} W'_n &= -4s \sum_{|\gamma|=n+2} \left(S^{\overline{22}\,\gamma} S_\gamma{}^{11} + S^{\overline{11}\,\gamma} S_\gamma{}^{22} \right) \\ &= -8s\,\mathrm{Re} \sum_{|\gamma|=n+2} S_{\gamma\,\overline{11}} S^\gamma{}_{22}. \end{aligned}$$

In the last sum, $S_{\gamma\,\overline{11}}$ vanishes unless γ is a permutation of $0\cdots022$ or $11n\cdots n$, while

$$S_{0\cdots022\,\overline{11}} = S_{22}{}^{11n\cdots n} = (-1)^{n+1} 4\cdot n!\, s,$$
$$S_{11n\cdots n\,\overline{11}} = S_{22}{}^{0\cdots022} = 2\,t.$$

Therefore $W'_n = (-1)^n 64\,(n+2)!\, s^2\, t$, so that $c_n = (-1)^n 64\,(n+2)!$.

We next prove (6.4). Since $I_2(A,C)$ contains no linear term, we have

$$I_2(A,C) = c_2\,(C^0)^2 + Q_2(A,C)$$

for a constant c_2. Restriction of this formula to $(A,C)=(0,C_t)$ yields

$$I_2(0,C_t) = c_2\,t^2 + O(t^3), \tag{6.9}$$

while, by the expression $\varphi = \mathcal{L}(0,t) = t\,|\zeta^0|^{-6}\mu^4/4!$,

$$I_2(0,C_t) = W_2' + O(t^3), \quad W_2' = \sum_{|\alpha|=6, |\beta|=2} S_{\alpha\overline{\beta}}\,S^{\overline{\beta}\,\alpha}.$$

Since $S_{\alpha\overline{0k}} = S^{\overline{2}\,\overline{k}\,\alpha} = 0$ for $k=0,1,2$ and any list α, we have

$$W_2' = \sum_{|\alpha|=6} S_{\alpha\,\overline{11}}\,S^{\overline{11}\,\alpha}.$$

In this sum, $S_{\alpha\,\overline{11}}$ vanishes unless α is a permutation of 001122, while

$$S_{001122\overline{11}} = S^{\overline{11}\,001122} = 4!\,t.$$

Thus $W_2' = 72\cdot 6!\,t^2$. This together with (6.9) yields $c_2 = 72\cdot 6!$. □

Remark 6.4. As a consequence of Lemma 6.2, we see that a CR invariant $I(A)$ of weight w can contain linear terms only when $w=n+1$ and that the linear part must be a constant multiple of $\Delta_x^{n+2} f_A(e_0)$. This fact is equivalent to Theorem 2.3 of Graham [G2].

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